MATH 556 - EXERCISES 4
These exercises are not for assessment

1. State whether each of the following functions defines an Exponential Family distribution. Where it is possible, write the distribution in the Exponential Family form, and find the natural (canonical) parameterization. If the function does not specify an Exponential Family distribution, explain why not.

   (a) The continuous $Uniform(\theta_1, \theta_2)$ distribution:

   $$f_X(x|\theta_1,\theta_2) = \frac{1}{\theta_2 - \theta_1} \qquad \theta_1 < x < \theta_2$$

   and zero otherwise, for parameters $\theta_1 < \theta_2$.

   (b) The distribution defined by

   $$f_X(x|\theta) = \frac{-1}{\log(1-\theta)} \frac{\theta^x}{x} \qquad x = 1, 2, 3, \ldots$$

   and zero otherwise, for parameter θ, where $0 < \theta < 1$.

   (c) The distribution defined by

   $$f_X(x|\phi,\lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left\{-\frac{\phi^2}{2\lambda}x + \phi - \frac{\lambda}{2x}\right\} \qquad x > 0$$

   and zero otherwise, for parameters $\phi, \lambda > 0$. Find the expectation of $1/X$ in terms of ϕ and λ.

2. For random variable X, consider a one parameter Exponential Family distribution in its natural parameterization with $k = 1$,

   $$f_X(x|\eta) = h(x)c^\star(\eta)\exp\{\eta\, t(x)\}$$

   and natural parameter space $\mathcal{H}$. Suppose that $\mathcal{H}$ is an open interval in $\mathbb{R}$, so that for every $\eta \in \mathcal{H}$, there exists an $\epsilon > 0$ such that

   $$\eta' \in \mathcal{H} \quad \text{if} \quad |\eta - \eta'| < \epsilon$$

   (a) Show that the natural parameter space $\mathcal{H}$ is a convex set

   $$\eta_1, \eta_2 \in \mathcal{H} \qquad \Longrightarrow \qquad \lambda\eta_1 + (1-\lambda)\eta_2 \in \mathcal{H}$$

   for $0 \leq \lambda \leq 1$.

   (b) Prove that the cumulant generating function of random variable $T = t(X)$ under the probability model f_X takes the form

   $$K_T(s) = \kappa(\eta + s) - \kappa(\eta)$$

   for $s \in (-h, h)$, some $h > 0$, where κ is some function to be identified.

   (c) Suppose that $\eta_1, \eta_2 \in \mathcal{H}$. Find the form of the log likelihood ratio, $\ell(x; \eta_1, \eta_2)$, where

   $$\ell(x;\eta_1,\eta_2) = \log\frac{f_X(x|\eta_1)}{f_X(x|\eta_2)}.$$

3. Suppose that $X_1, \ldots, X_r$ are independent random variables such that, for each i, $X_i \sim N(\mu_i, 1)$, for fixed constants $\mu_1, \ldots, \mu_r$.

   (a) Find the mgf of random variable Y defined by
   $$Y = \sum_{i=1}^{r} X_i^2.$$

   (b) Find the skewness of Y, ς, where
   $$\varsigma = \frac{\mathbb{E}_{f_Y}[(Y-\mu)^3]}{\sigma^3}$$
   where μ and σ^2 are the expectation and variance of f_Y.

4. Consider the three-level hierarchical model:
   $$\begin{aligned} &\text{LEVEL 3}: && \lambda > 0, r \in \{1, 2, \ldots\} && \text{Fixed parameters} \\ &\text{LEVEL 2}: && N \sim Poisson(\lambda) && \\ &\text{LEVEL 1}: && X|N = n \sim Gamma(n + r/2, 1/2) && \end{aligned}$$
   Find

   (a) The expectation of X, $\mathbb{E}_{f_X}[X]$,
   (b) The mgf of X, $M_X(t)$.

5. Consider the three-level hierarchical model:
   $$\begin{aligned} &\text{LEVEL 3}: && \mu \in \mathbb{R}, \tau, \sigma > 0 && \text{Fixed parameters} \\ &\text{LEVEL 2}: && M \sim Normal(\mu, \tau^2) && \\ &\text{LEVEL 1}: && X_1, X_2|M = m \sim Normal(m, \sigma^2) && \end{aligned}$$
   where X_1 and X_2 are conditionally independent given M, denoted
   $$X_1 \perp X_2 \mid M.$$
   Using the law of iterated expectation, find the (marginal) covariance and correlation between X_1 and X_2. Are X_1 and X_2 (marginally) independent ? Justify your answer.

6. In a branching process model, the total number of individuals in successive generations are random variables $S_0, S_1, S_2, \ldots$. Suppose that, in the passage from generation i to generation $i+1$, each of the s_i individuals observed in generation i gives rise to N_{ij} offspring for $j = 1, \ldots, s_i$ according to a pmf with corresponding pgf G_N.

   In addition to the production of offspring, suppose that at each generation, immigration into the population is allowed, and that at generation i, K_i immigrants enter the population to go forward to the $i+1$st generation, so that
   $$S_{i+1} = \sum_{j=1}^{s_i} N_{ij} + K_i$$
   where $K_0, K_1, K_2, \ldots$ are iid random variables, with pgf G_K, that are independent of all N_{ij}.

   Find the pgf of S_{i+1} in terms of the pgf of random variable S_i and G_K.