MATH 556 - EXERCISES 2

These exercises are not for assessment

1. The radius of a circle, R, is a continuous random variable with density function given by
$$f_R(r) = 6r(1-r) \qquad 0 < r < 1$$
and zero otherwise. Find the density functions of the circumference and the area of the circle.

2. Suppose that X and Y are continuous random variables with joint pdf given by
$$f_{X,Y}(x,y) = cx(1-y) \qquad 0 < x < 1,\ 0 < y < 1$$
and zero otherwise for some constant c. Are X and Y independent random variables ?

   Find the value of c, and, for the set $A \equiv \{(x,y) : 0 < x < y < 1\}$, the probability
$$P[X < Y] = \iint\limits_A f_{X,Y}(x,y)\, dxdy$$

3. Suppose that X and Y are continuous random variables with joint pdf given by
$$f_{X,Y}(x,y) = \frac{1}{2x^2 y} \qquad 1 \le x < \infty, 1/x \le y \le x$$
and zero otherwise. Derive
   (i) the marginal pdf of X and Y
   (ii) the conditional pdf of X given $Y = y$, and the conditional pdf of Y given $X = x$.
   (iii) the expectation of Y, $E_{f_Y}[Y]$.

4. Suppose that X and Y have joint pdf that is constant with support $\mathbb{X}^{(2)} \equiv (0,1) \times (0,1)$.
   (i) Find the marginal pdf of random variables $U = X/Y$ and $V = -\log(XY)$, stating clearly the range of the transformed random variable in each case.
   (ii) Find the pdf and cdf of $Z = X - Y$.

5. Suppose that X is a random variable with pmf/pdf f_X and mgf M_X, where for some $h > 0$,
$$M_X(t) = E_{f_X}[e^{tX}] = \int e^{tx}\, dF_X(t) \qquad\qquad -h < t < h$$
The cumulant generating function of X, K_X, is defined by $K_X(t) = \log M_X(t)$. Show that
$$\frac{d}{dt}\left\{K_X(t)\right\}_{t=0} = E_{f_X}[X] \qquad\qquad \frac{d^2}{dt^2}\left\{K_X(t)\right\}_{t=0} = Var_{f_X}[X]$$

6. Suppose that X and Y are independent $Normal(0,1)$ random variables.
   (i) Let random variable U be defined by $U = X/Y$. Find the pdf of U.
   (ii) Suppose now that S is a random variable, independent of X and Y, where $S \sim Gamma(\nu/2, 1/2)$ where ν is a positive integer. Find the pdf of random variable T defined by
$$T = \frac{X}{\sqrt{S/\nu}}$$
   (iii) Suppose now that the joint pdf of random variables X and Y is specified via the conditional density $f_{X|Y}$ and the marginal density f_Y as
$$f_{X|Y}(x|y) = \sqrt{\frac{y}{2\pi}} \exp\left\{-\frac{yx^2}{2}\right\} \qquad x \in \mathbb{R} \qquad\qquad f_Y(y) = \frac{(\nu/2)^{(\nu/2)}}{\Gamma(\nu/2)} y^{\nu/2-1} e^{-\nu y/2} \qquad y > 0$$
and zero otherwise, where ν is a positive integer. Find the marginal pdf of X.