MATH 556 - ASSIGNMENT 4: SOLUTIONS

1 (a) Directly from the notes: by Lyapunov's inequality

$$\mathbb{E}[\, |X_n - X|^s \,]^{1/s} \leq \mathbb{E}[\, |X_n - X|^r \,]^{1/r}$$

so that

$$\mathbb{E}[\, |X_n - X|^s \,] \leq \mathbb{E}[\, |X_n - X|^r \,]^{s/r} \longrightarrow 0$$

as $n \longrightarrow \infty$, as $s < r$. Thus

$$\mathbb{E}[\, |X_n - X|^s \,] \longrightarrow 0$$

and $X_n \overset{s^{th}}{\longrightarrow} X$.

2 MARKS

For the given sequence, if $r > s \geq 1$,

$$\mathbb{E}[|X_n|^s] = \frac{n^s}{n^{(r+s)/2}} = n^{(s-r)/2} \longrightarrow 0$$

whilst

$$\mathbb{E}[|X_n|^r] = \frac{n^r}{n^{(r+s)/2}} = n^{(r-s)/2} \longrightarrow \infty$$

as $n \longrightarrow \infty$.

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(b) Directly from the notes: suppose $X_n \overset{r=1}{\longrightarrow} X$, and let $\epsilon > 0$. Then, using an argument similar to Chebychev's Lemma,

$$\mathbb{E}[\, |X_n - X| \,] \geq \mathbb{E}[\, |X_n - X| \, \mathbb{I}_{\{(\epsilon,\infty)\}} \, (|X_n - X|) \,] \geq \epsilon P[|X_n - X| > \epsilon].$$

as $|X_n - X| > \epsilon$ on $\{(\epsilon, \infty)\}$. Taking limits as $n \longrightarrow \infty$,

$$X_n \overset{r=1}{\longrightarrow} X \qquad \Longrightarrow \qquad \mathbb{E}[\, |X_n - X|^r \,] \longrightarrow 0$$

so therefore, also, as $n \longrightarrow \infty$

$$P[|X_n - X| > \epsilon] \longrightarrow 0 \qquad \therefore \qquad X_n \overset{p}{\longrightarrow} X.$$

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For the given sequence, we have for $\epsilon > 0$,

$$P[|X_n| > \epsilon] = 1/n^2 \longrightarrow 0 \qquad \therefore \qquad X_n \overset{p}{\longrightarrow} 0$$

However

$$\mathbb{E}[|X_n|] = n^\alpha / n^2 = n^{\alpha - 2} \longrightarrow \begin{cases} 0 & \alpha < 2 \\ 1 & \alpha = 2 \\ \infty & \alpha > 2 \end{cases}$$

Hence if $\alpha \geq 2$, $X_n \overset{p}{\longrightarrow} 0$, but $\{X_n\}$ does not converge in mean to zero.

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2 Let $V_n = \log T_n$. Then

$$V_n = \frac{1}{n}\sum_{i=1}^{n} \log X_i$$

Now, if $U \sim Uniform(0,1)$, then $X = -\log U \sim Exponential(1)$ with $\mathbb{E}_{f_X}[X] = \text{Var}_{f_X}[X] = 1$ so that, by properties of expectations,

$$\mathbb{E}_{f_{V_n}}[V_n] = -1 \qquad\qquad \text{Var}_{f_{V_n}}[V_n] = \frac{1}{n}$$

Hence, by the Central Limit Theorem,

$$\sqrt{n}(V_n + 1) \xrightarrow{d} Z \sim \mathcal{N}(0,1)$$

Now let $g(x) = e^x$, so that $\dot{g}(x) = e^x$. By the Delta Mathod

$$\sqrt{n}(e^{V_n} - e^{-1}) \xrightarrow{d} Z \sim \mathcal{N}(0, e^{-2})$$

but as $e^{V_n} = T_n$, we have

$$\sqrt{n}(T_n - e^{-1}) \xrightarrow{d} Z \sim \mathcal{N}(0, e^{-2})$$

so that, approximately for large n

$$T_n \stackrel{\cdot}{\sim} \mathcal{N}(e^{-1}, e^{-2}/n)$$

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```
set.seed(10101)
N<-5000; n<-100
Xmat<-matrix(runif(N*n),nrow=N); Tvec<-apply(Xmat,1,prod)^(1/n)
htitle<-expression(paste(``Histogram of '',T[n]))
hist(Tvec,br=seq(0.2,0.6,by=0.01),xlab=expression(T[n]),main=htitle,ylim=range(0,600))
xv<-seq(0.2,0.6,length=1001); yv<-dnorm(xv,exp(-1),sqrt(exp(-2)/n))
lines(xv,yv*N*0.01,col=``red'',lwd=2)
```

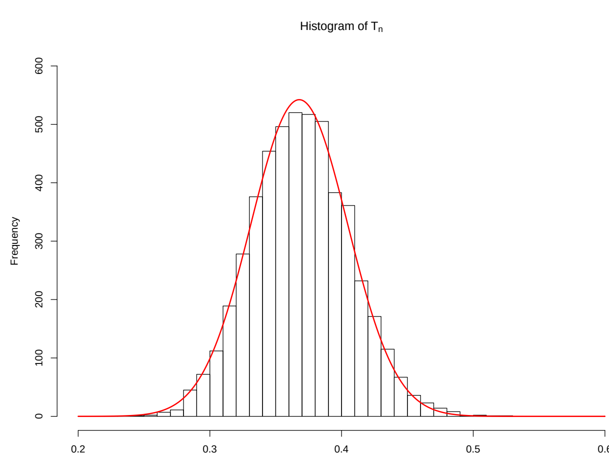

Figure 1: Histogram of T_n with approximation (red)

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