MATH 556 - ASSIGNMENT 1: SOLUTIONS

1 The distribution of any discrete random variable, X, can be written as a linear combination of a countable number of point mass measures

$$P_X(B) = \sum_{i=1}^{\infty} p_i \delta_{x_i}(B)$$

where $x_1, x_2, \ldots$ are the countable set of values at which f_X is non-zero, and $p_1, p_2 \ldots$ are the constants that record the probability masses at $x_1, x_2, \ldots$ such that $p_i = f_X(x_i)$.

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2 Clearly $F_X(x)$ exhibits the required properties of a cdf as

(i) $F_X(x)$ is non-decreasing, as both $F_1(x)$ and $F_2(x)$ are non-decreasing, and $F_1(c) < F_2(c)$.

(ii) We have the correct limit behaviour

$$\lim_{x \longrightarrow -\infty} F_X(x) = \lim_{x \longrightarrow -\infty} F_1(x) = 0 \qquad \lim_{x \longrightarrow \infty} F_X(x) = \lim_{x \longrightarrow \infty} F_2(x) = 1$$

(iii) We have right-continuity, as both $F_1(x)$ and $F_2(x)$ are right-continuous, and

$$\lim_{x \longrightarrow c^+} F_X(x) = F_X(c)$$

The corresponding probability measure can be defined for sets $B \in \mathscr{B}$ by

$$P_X(B) = P_1(B \cap (-\infty, c)) + \delta_c(B)(F_2(c) - F_1(c)) + P_2(B \cap (c, \infty)).$$

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3 We have from the formula sheet

$$h_X(x) = \frac{f_X(x)}{1 - F_X(x-1)} = \frac{(1-\theta)^{x-1}\theta}{1 - (1 - (1-\theta)^{x-1})} = \theta$$

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4 (a) This **is not** a cdf: we have for example

$$\frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_2} = -2e^{-(x_1 + 2x_2)} < 0.$$

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(b) This **is** a cdf: it satisfies all of the requirements of a cdf, and

$$f_{X_1,X_2}(x_1, x_2) = \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_2} = \begin{cases} e^{-x_2} & 0 \le x_1 \le x_2 < \infty \\ 0 & \text{otherwise} \end{cases}$$

at all points where F is differentiable, where, for all (x_1, x_2)

$$\int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_{X_1,X_2}(t_1, t_2)\, dt_2 dt_1 = F(x_1, x_2).$$

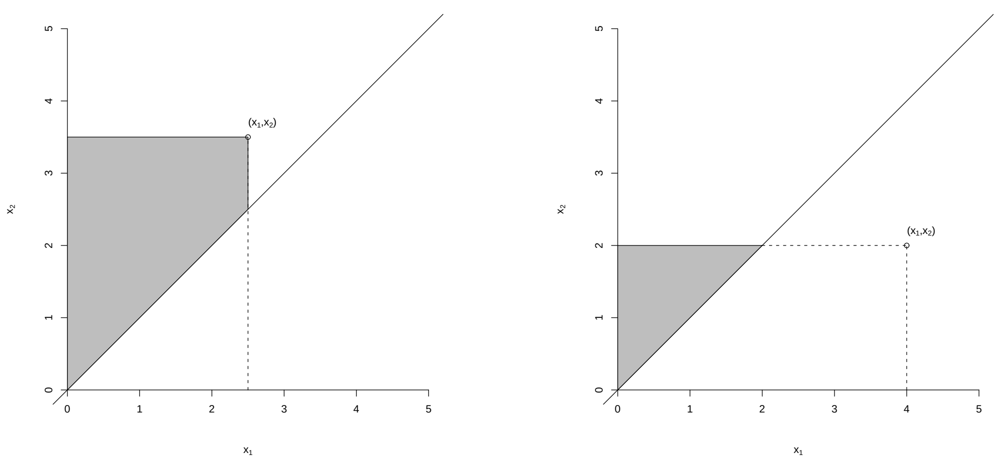
To see this, note that for $0 \leq x_1 \leq x_2 < \infty$ we have

$$
\begin{aligned}
\int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_{X_1,X_2}(t_1,t_2)\, dt_2 dt_1 &= \int_0^{x_1} \int_{t_1}^{x_2} e^{-t_2}\, dt_2 dt_1 \\
&= \int_0^{x_1} \left(e^{-t_1} - e^{-x_2} \right)\, dt_1 \\
&= \left[-e^{-t_1} - t_1 e^{-x_2} \right]_0^{x_1} \\
&= 1 - e^{-x_1} - x_1 e^{-x_2}
\end{aligned}
$$

(see Figure 1(a)) whereas for $0 \leq x_2 \leq x_1 < \infty$ we have

$$
\begin{aligned}
\int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_{X_1,X_2}(t_1,t_2)\, dt_2 dt_1 &= \int_0^{x_2} \int_{t_1}^{x_2} e^{-t_2}\, dt_2 dt_1 \\
&= \int_0^{x_2} \left(e^{-t_1} - e^{-x_2} \right)\, dt_1 \\
&= \left[-e^{-t_1} - t_1 e^{-x_2} \right]_0^{x_2} \\
&= 1 - e^{-x_2} - x_2 e^{-x_2}
\end{aligned}
$$

(see Figure 1(b)) as required.



(a) $0 \leq x_1 \leq x_2 < \infty$

(b) $0 \leq x_2 \leq x_1 < \infty$

Figure 1: Shaded region indicates region of integration: note that the joint pdf is only non-zero on the region $0 \leq x_1 \leq x_2 < \infty$

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