MATH 556 - ASSIGNMENT 4

To be handed in not later than 5pm, 27th November 2008.
Please hand in during lectures, to Burnside 1235, or to the Mathematics Office Burnside 1005

1 (a) Let $r > s \geq 1$. Prove that, for **convergence in mean**,

$$X_n \xrightarrow{r^{th}} X \qquad \Longrightarrow \qquad X_n \xrightarrow{s^{th}} X.$$

that is, convergence in rth mean implies convergence in sth mean. Show that the converse does not hold in general by using the following counterexample: let $\{X_n\}$ be defined by

$$X_n = \begin{cases} n & \text{with probability } n^{-(r+s)/2} \\ 0 & \text{with probability } 1 - n^{-(r+s)/2} \end{cases}$$

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(b) Prove that

$$X_n \xrightarrow{r=1} X \qquad \Longrightarrow \qquad X_n \xrightarrow{p} X.$$

Show that the converse does not hold in general by using the following counterexample: let $\{X_n\}$ be defined by

$$X_n = \begin{cases} n^\alpha & \text{with probability } 1/n^2 \\ 0 & \text{with probability } 1 - 1/n^2 \end{cases}$$

for constant α.

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2 Consider the following two results

- *The Central Limit Theorem:* Suppose $\{X_n\}$ are i.i.d. random variables with mgf M_X, with expectation μ and variance σ^2, both finite. Let the random variable Z_n be defined by

$$Z_n = \frac{\sum\limits_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}} = \frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} \qquad \text{where} \qquad \overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Then, as $n \longrightarrow \infty$, $Z_n \xrightarrow{d} Z \sim \mathcal{N}(0,1)$.

- *The Delta Method*: Consider sequence of random variables $\{X_n\}$ such that

$$\sqrt{n}(X_n - \mu) \xrightarrow{d} X.$$

Suppose that $g(.)$ is a function such that first derivative $\dot{g}(.)$ is continuous in a neighbourhood of μ, with $\dot{g}(\mu) \neq 0$. Then

$$\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{d} \dot{g}(\mu)X.$$

In particular, if

$$\sqrt{n}(X_n - \mu) \xrightarrow{d} X \sim \mathcal{N}(0, \sigma^2).$$

then

$$\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{d} \dot{g}(\mu)X \sim \mathcal{N}(0, \{\dot{g}(\mu)\}^2\sigma^2).$$

Using these two results, find a large sample approximation (n large but finite) to the distribution of the statistic

$$T_n = \left\{ \prod_{i=1}^{n} X_i \right\}^{1/n}$$

if $X_1, X_2, \ldots,$ are independent $Uniform(0,1)$ random variables.

Verify your result by simulating 5000 copies of T_n for $n = 100$, and plotting a histogram and your approximate density.

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