

MATH 545: INTRODUCTION TO TIME SERIES ANALYSIS

WINTER 2025

SPECTRAL AND FREQUENCY DOMAIN CONCEPTS

The ACVF of a (zero-mean) stationary process $\{X_t\}$

$$\gamma_X(h) = \mathbb{E}[X_t X_{t+h}]$$

is a symmetric, non-negative definite function of the discrete argument $h = 0, \pm 1, \pm 2, \dots$. We exploit the fact that $\gamma_X(h)$ has a unique representation in terms of *frequency contributions*. Let $\{X_t\}$ be stationary and zero-mean, with ACVF $\gamma_X(\cdot)$. Then the *spectral density function* (SDF) of $\{X_t\}$ is the function $f_X(\cdot)$ defined by

$$f_X(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\lambda h} \gamma_X(h) \quad \lambda \in \mathbb{R}$$

provided this sum is convergent; if

$$\sum_{h=-\infty}^{\infty} |\gamma_X(h)| < \infty.$$

then $f_X(\lambda)$ is well-defined, as $|e^{-i\lambda h}| = 1$, so

$$\sum_{h=-\infty}^{\infty} |e^{-i\lambda h} \gamma_X(h)| \leq \sum_{h=-\infty}^{\infty} |e^{-i\lambda h}| |\gamma_X(h)| = \sum_{h=-\infty}^{\infty} |\gamma_X(h)| < \infty.$$

although the absolute summability is not a necessary condition. Also, $f_X(\lambda)$ is *periodic* as

$$e^{-i\lambda h} = \cos(\lambda h) - i \sin(\lambda h)$$

so we can restrict attention to $\lambda \in (-\pi, \pi]$. We have that

(i) $f_X(\lambda) = f_X(-\lambda)$ and

$$\begin{aligned} f_X(\lambda) &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\lambda h} \gamma_X(h) = \frac{1}{2\pi} \left[\gamma_X(0) + \sum_{h=1}^{\infty} (e^{-i\lambda h} + e^{i\lambda h}) \gamma_X(h) \right] && \text{as } \gamma(-h) = \gamma(h) \\ &= \frac{1}{2\pi} \left[\gamma_X(0) + \sum_{h=1}^{\infty} 2 \cos(\lambda h) \gamma_X(h) \right] \\ &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \cos(\lambda h) \gamma_X(h) && \text{as } \cos(-\lambda h) = \cos(\lambda h) \end{aligned}$$

(ii) $f_X(\lambda) \geq 0$, $-\pi < \lambda \leq \pi$.

(iii) Inversion formula:

$$\int_{-\pi}^{\pi} e^{ih\lambda} f_X(\lambda) d\lambda = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \sum_{k=-\infty}^{\infty} e^{i\lambda(h-k)} \gamma_X(k) \right\} d\lambda = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_X(k) \left\{ \int_{-\pi}^{\pi} e^{i\lambda(h-k)} d\lambda \right\} = \gamma_X(h).$$

The last result follows as if $k \neq h$, setting $j = h - k$

$$\int_{-\pi}^{\pi} e^{i\lambda(h-k)} d\lambda = \int_{-\pi}^{\pi} [\cos(\lambda j) + i \sin(\lambda j)] d\lambda = \int_{-\pi}^{\pi} \cos(\lambda j) d\lambda + i \int_{-\pi}^{\pi} \sin(\lambda j) d\lambda = 0.$$

Note also that

$$\gamma_X(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f_X(\lambda) d\lambda = \int_{-\pi}^{\pi} [\cos(h\lambda) + i \sin(h\lambda)] f_X(\lambda) d\lambda = \int_{-\pi}^{\pi} \cos(h\lambda) f_X(\lambda) d\lambda$$

as $\sin(x)$ is an odd function.

There is a 1-1 correspondence between ACVFs and spectral density functions; if

(i) $f_X(\lambda) \geq 0$, for all $\lambda \in (0, \pi]$.

(ii) For $h = 0, \pm 1, \pm 2, \dots$, if

$$\gamma_X(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f_X(\lambda) d\lambda$$

then f_X is the SDF of $\{X_t\}$ with ACVF $\gamma_X(h)$. For a generalization, a function $\gamma(\cdot)$, $\gamma : \mathbb{Z} \rightarrow \mathbb{R}$, is an autocovariance function if and only if there exists a right continuous, non-decreasing bounded function F on $[-\pi, \pi]$ with $F(-\pi) = 0$, and for all $h \in \mathbb{Z}$

$$\gamma(h) = \int_{(-\pi, \pi]} e^{ih\lambda} dF(\lambda).$$

As the corresponding series $\{\gamma(h)\}$ is real-valued, F satisfies

$$\int_{(a, b]} dF(x) = \int_{(-b, -a]} dF(x)$$

- The notation $dF(\lambda)$ represents the infinitesimal change in F at λ .
- F is a *generalized distribution function* as $F(\lambda)/F(\pi)$ is a distribution function on $[-\pi, \pi]$.
- F is termed the *spectral distribution function*; it can be purely discrete, absolutely continuous, or mixed. If F is absolutely continuous, then the *spectral density function* f is defined by

$$F(\lambda) = \int_{-\pi}^{\lambda} f(x) dx \quad \lambda \in (-\pi, \pi]$$

Harmonic Process: Suppose

$$X_t = U \cos(\omega t) + V \sin(\omega t)$$

where U and V are uncorrelated, with

$$\mathbb{E}[U] = \mathbb{E}[V] = 0 \quad \text{Var}[U] = \text{Var}[V] = 1$$

Then by a previous result

$$\gamma_X(h) = \cos(\omega h) \quad h \in \mathbb{Z}$$

so that

$$\sum_{h=-\infty}^{\infty} |\gamma_X(h)| = \sum_{h=-\infty}^{\infty} |\cos(\omega h)| = \infty$$

and $|\gamma_X(h)|$ does not converge to any limit. However, let

$$F_X(\lambda) = \begin{cases} 0 & \lambda < -\omega \\ \frac{1}{2} & -\omega \leq \lambda < \omega \\ 1 & \omega \leq \lambda \end{cases}$$

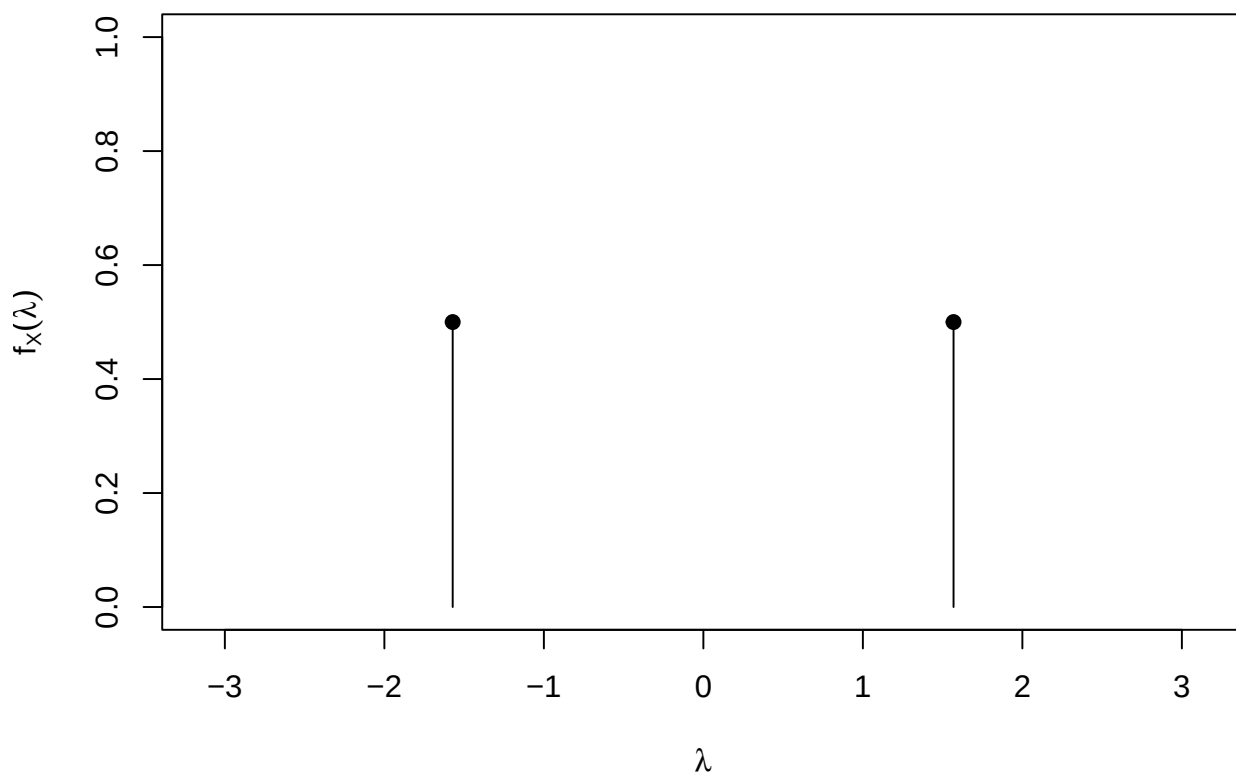
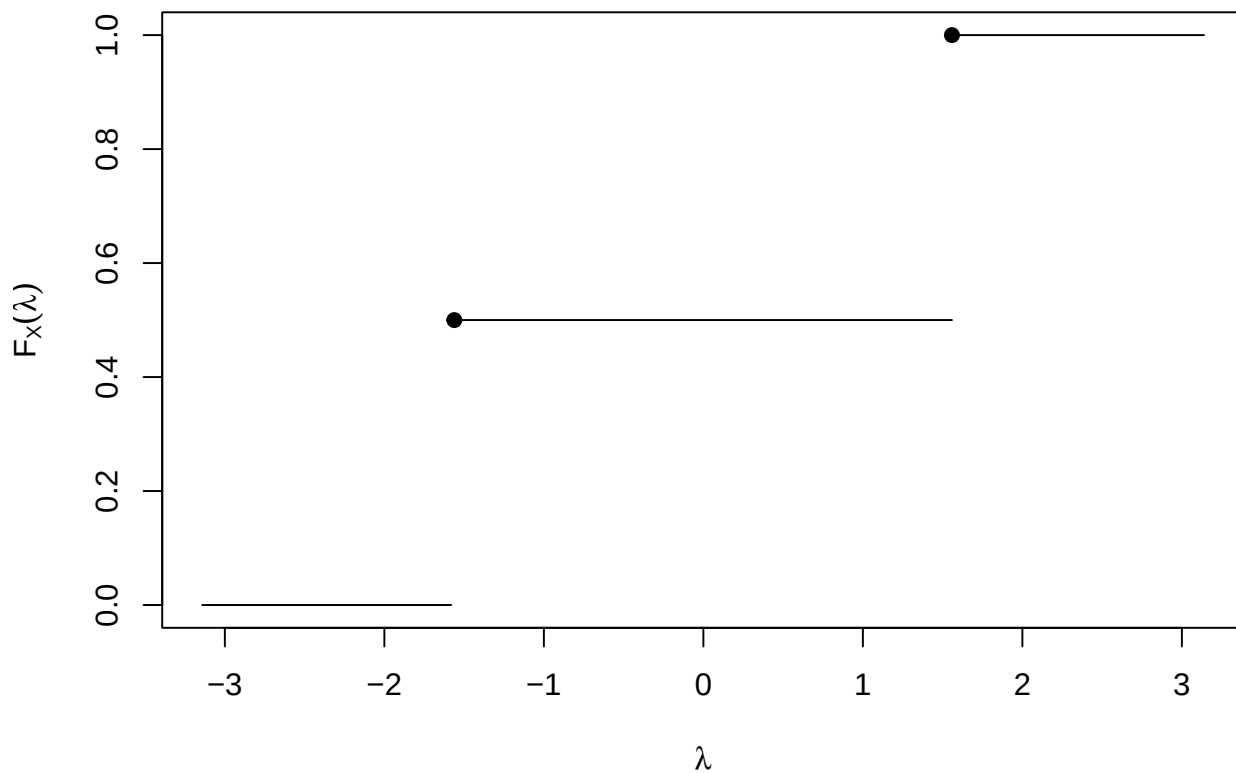
and define

$$f_X(\lambda) = \begin{cases} \frac{1}{2} & \lambda = -\omega \\ \frac{1}{2} & \lambda = \omega \\ 0 & \text{otherwise} \end{cases}$$

with the interpretation

$$\int_{(-\pi, \pi)} e^{ih\lambda} dF_X(\lambda) = \frac{1}{2}e^{-ih\omega} + \frac{1}{2}e^{ih\omega} = \cos(h\omega)$$

Thus $F_X(\lambda)$ is a right continuous step function with steps at $\pm\omega$, and $f_X(\lambda)$ takes the value of $1/2$ at $\pm\omega$. The variation in X has a characteristic frequency ω that indicates its periodicity.



Superposition of harmonic processes: Suppose

$$X_t = \sum_{j=1}^k [U_j \cos(\omega_j t) + V_j \sin(\omega_j t)]$$

where $U_1, \dots, U_k$ and $V_1, \dots, V_k$ are uncorrelated, with

$$\mathbb{E}[U_j] = \mathbb{E}[V_j] = 0 \quad \text{Var}[U_j] = \text{Var}[V_j] = \sigma_j^2$$

and where $0 < \omega_1 < \dots < \omega_k < \pi$. Then

$$\gamma_X(h) = \sum_{j=1}^k \sigma_j^2 \cos(\omega_j h) \quad h \in \mathbb{Z}.$$

Then

$$F_X(\lambda) = \sum_{j=1}^k \sigma_j^2 F_j(\lambda)$$

where

$$F_j(\lambda) = \begin{cases} 0 & \lambda < -\omega_j \\ \frac{1}{2} & -\omega_j \leq \lambda < \omega_j \\ 1 & \omega_j \leq \lambda \end{cases}$$

Note that

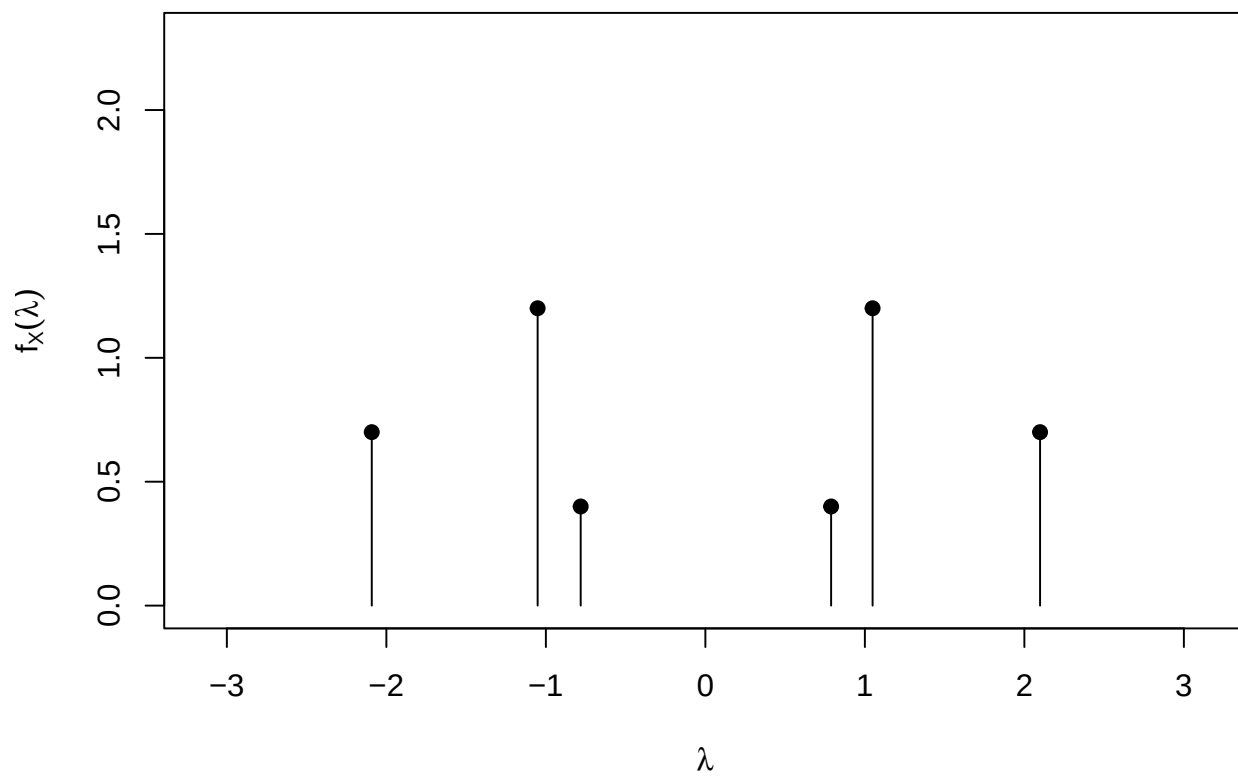
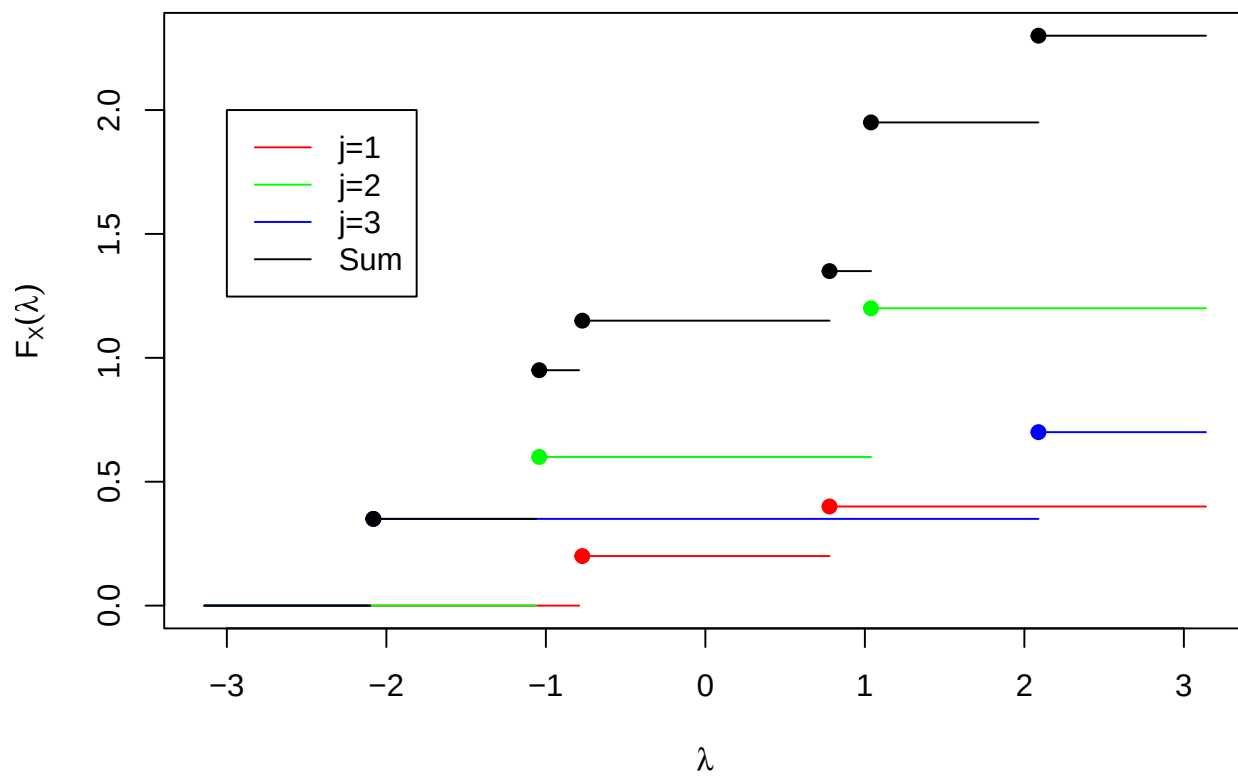
$$\text{Var}[X_t] = \gamma_X(0) = \sum_{j=1}^k \sigma_j^2$$

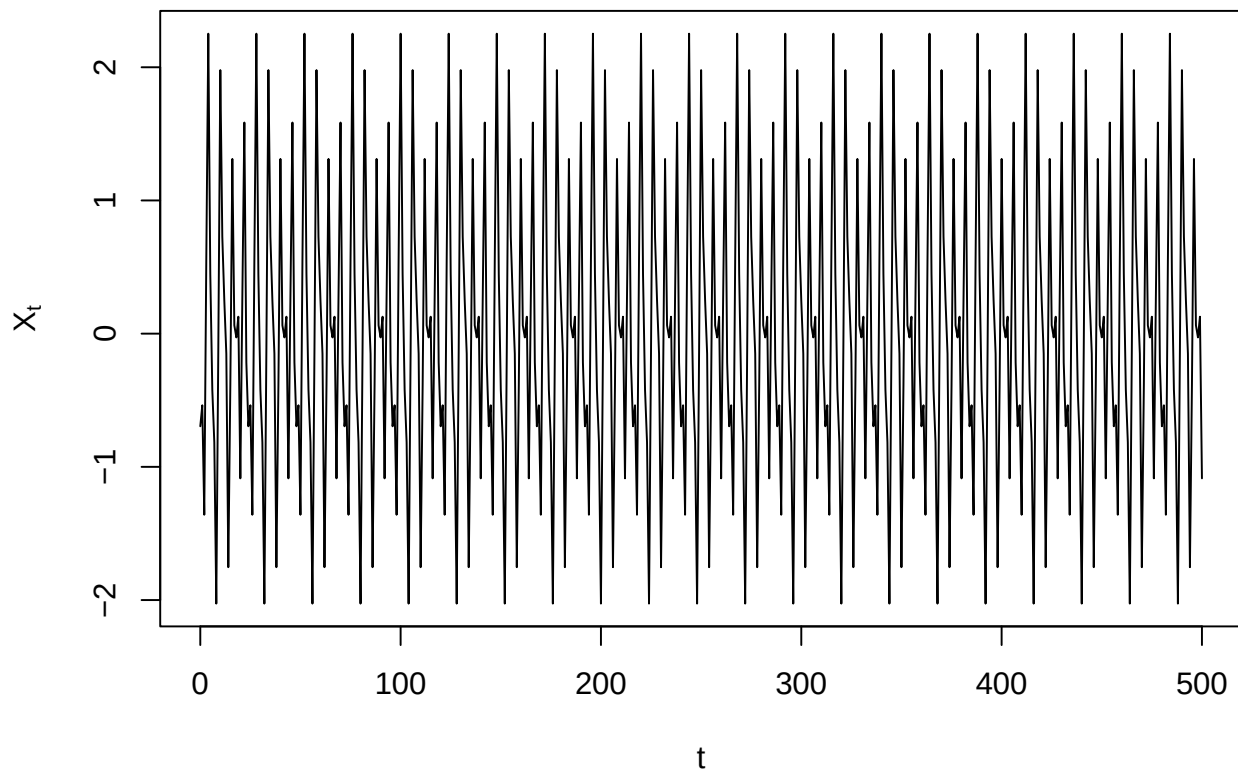
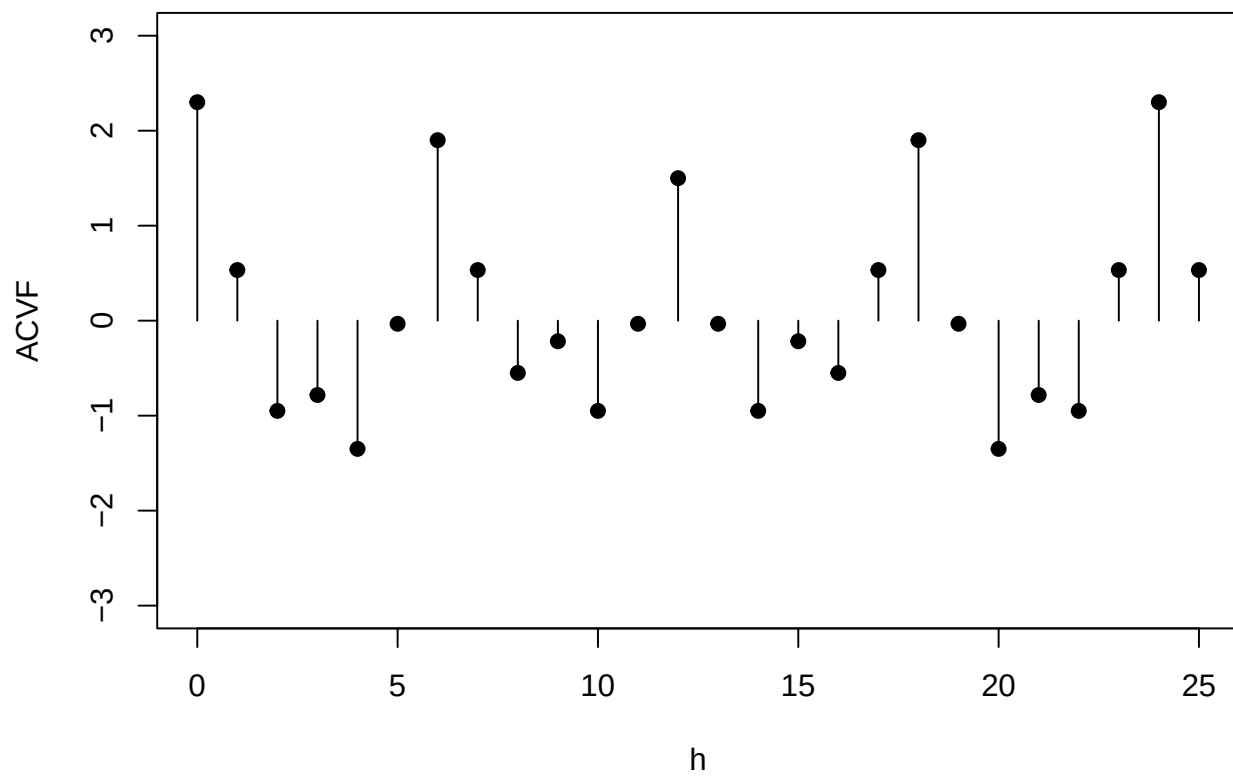
Suppose $k = 3$, with

- $j = 1: \omega_1 = \pi/4, \sigma_1^2 = 0.4$
- $j = 2: \omega_2 = \pi/3, \sigma_2^2 = 1.2$
- $j = 3: \omega_3 = 2\pi/3, \sigma_3^2 = 0.7$

```
lam<-seq(-pi,pi,by=0.01)
omega<-c(pi/4,pi/3, 2*pi/3)
sigsq<-c(0.4,1.2,0.7)
hvec<-c(0:25)
gammah<-hvec*0
for(j in 1:3){
  gammah<-gammah+sigsq[j]*cos(omega[j]*hvec)
}
acvf<-cbind(hvec,gammah)
rownames(acvf)<-paste('h=',hvec,sep='')
colnames(acvf)<-c('h','ACVF')
acvf[1:10,2]
```

##	h=0	h=1	h=2	h=3	h=4	h=5
##	2.30000000	0.53284271	-0.95000000	-0.78284271	-1.35000000	-0.03284271
##	h=6	h=7	h=8	h=9		
##	1.90000000	0.53284271	-0.55000000	-0.21715729		





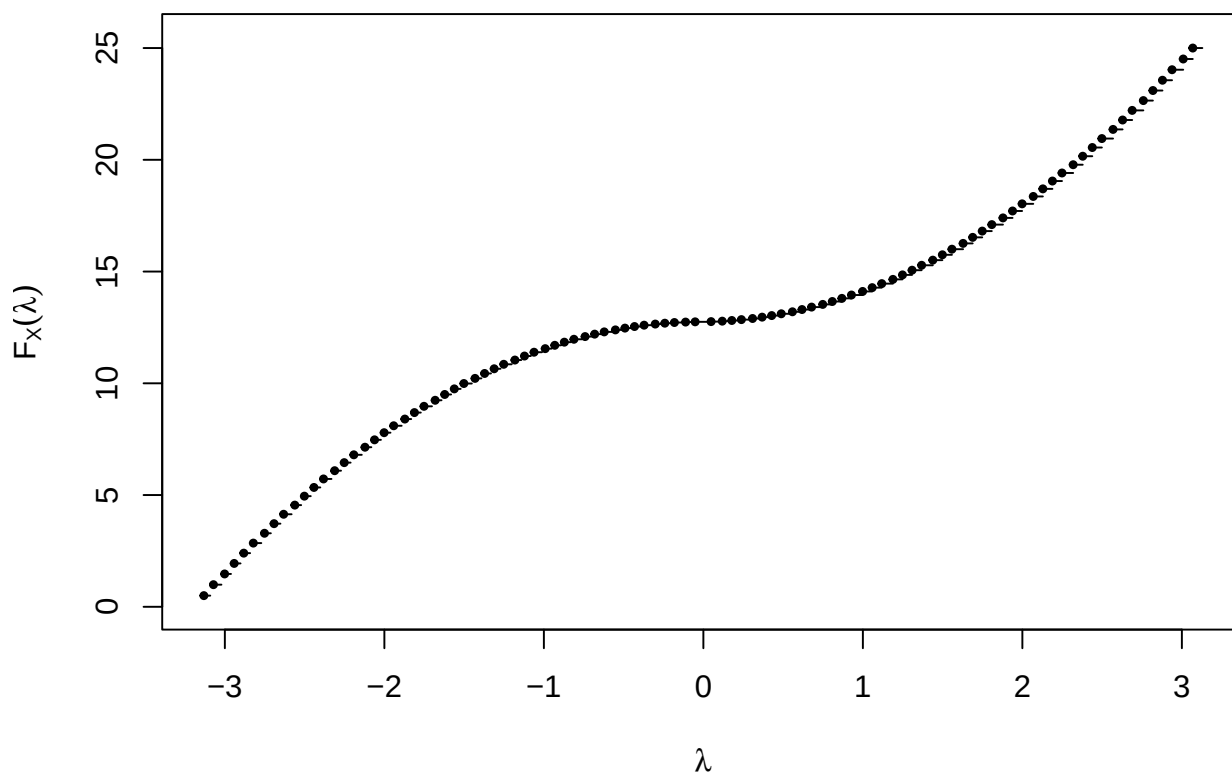
Now suppose $k = 50$, with

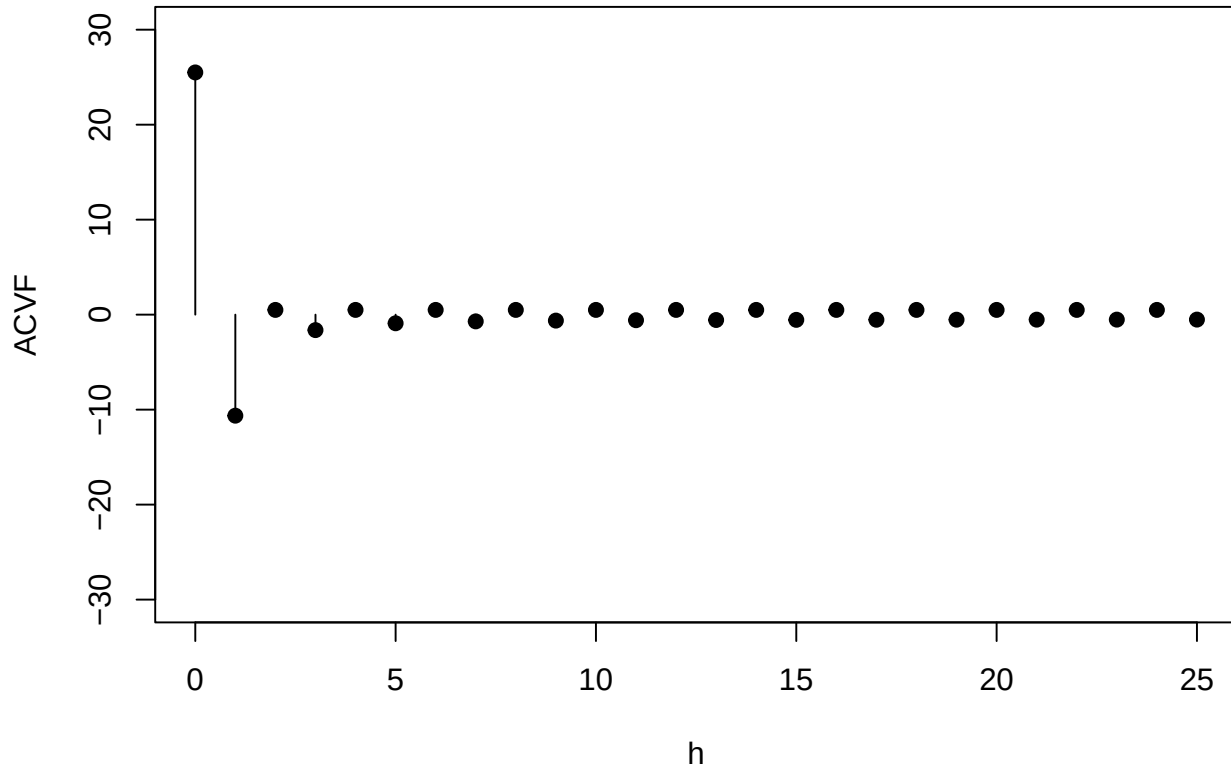
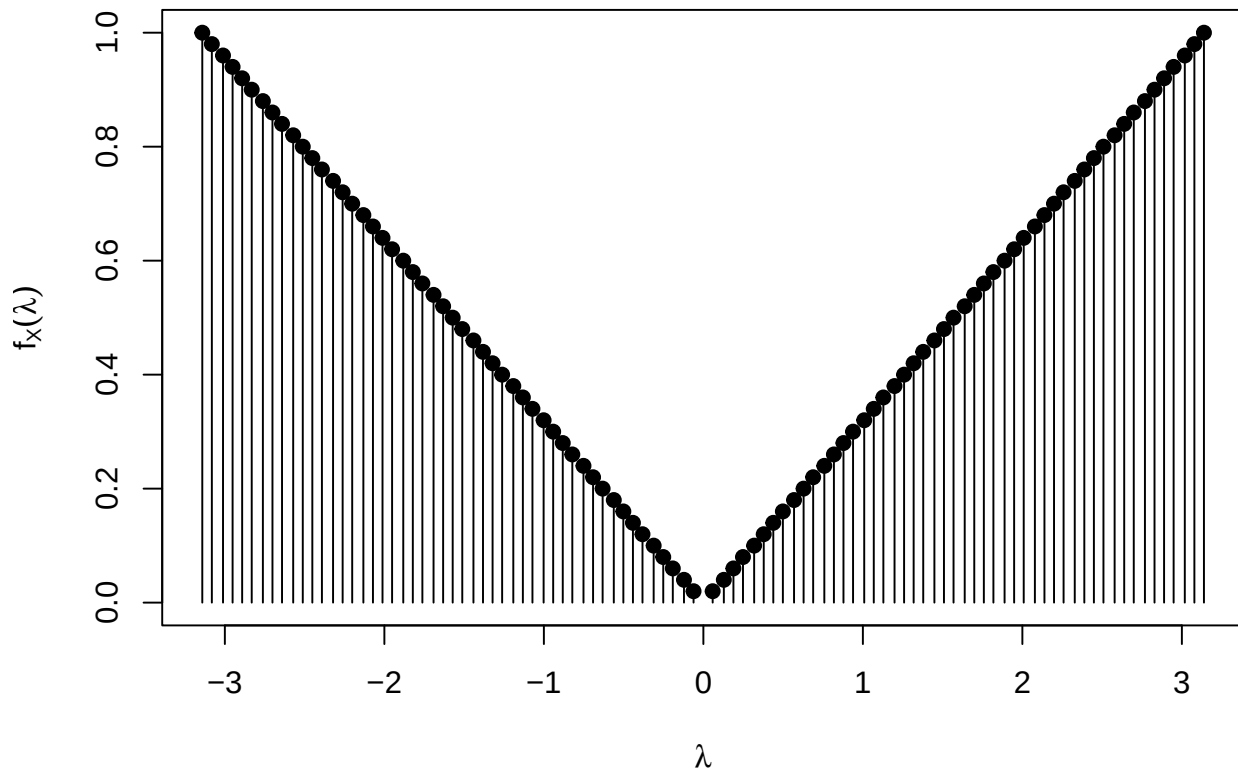
- $\omega_j = j\pi/k$,
- $\sigma_j^2 = j/k$

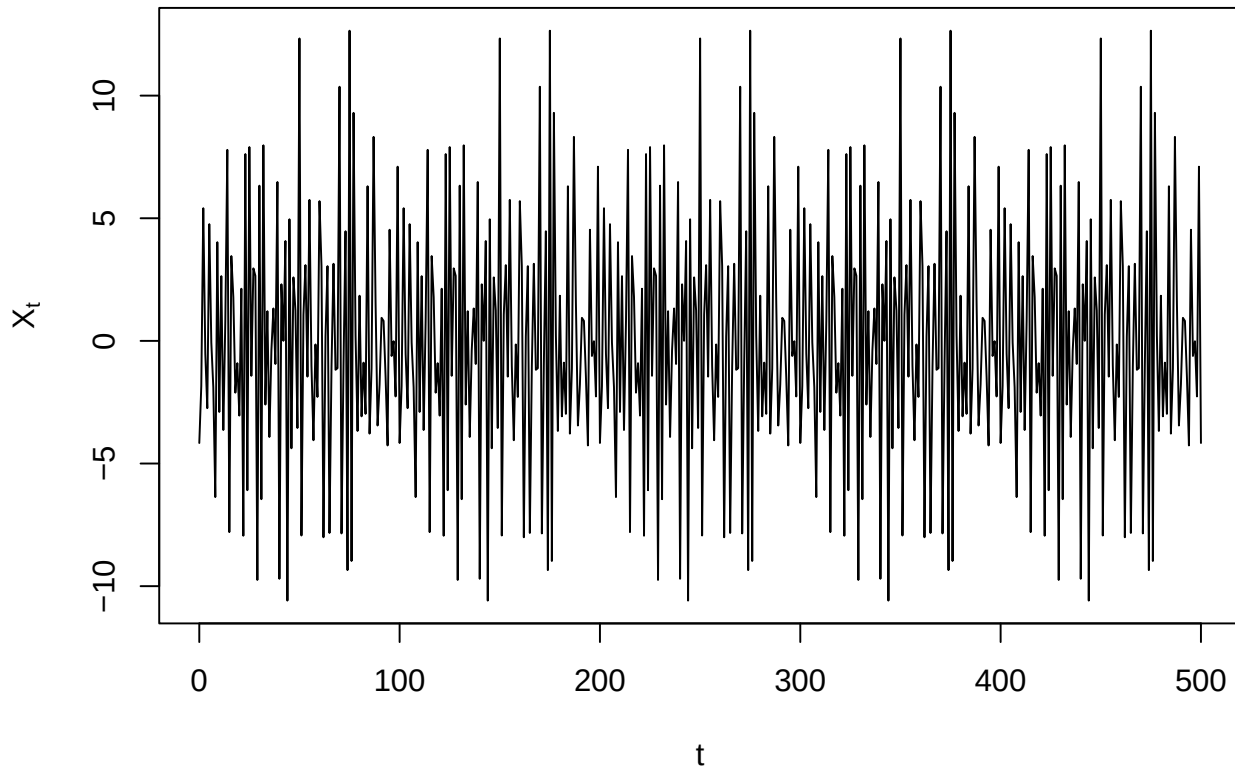
for $j = 1, \dots, k$.

```
k<-50
lam<-seq(-pi,pi,by=0.01)
omega<-c(1:k)*pi/k
sigsq<-c(1:k)/k
hvec<-c(0:25)
gammah<-hvec*0
for(j in 1:k){
  gammah<-gammah+sigsq[j]*cos(omega[j]*hvec)
}
acvf<-cbind(hvec,gammah)
rownames(acvf)<-paste('h=',hvec,sep='')
colnames(acvf)<-c('h','ACVF')
acvf[1:10,2]
```

##	h=0	h=1	h=2	h=3	h=4	h=5
##	25.5000000	-10.6354524	0.5000000	-1.6291302	0.5000000	-0.9086346
##	h=6	h=7	h=8	h=9		
##	0.5000000	-0.7101437	0.5000000	-0.6284752		







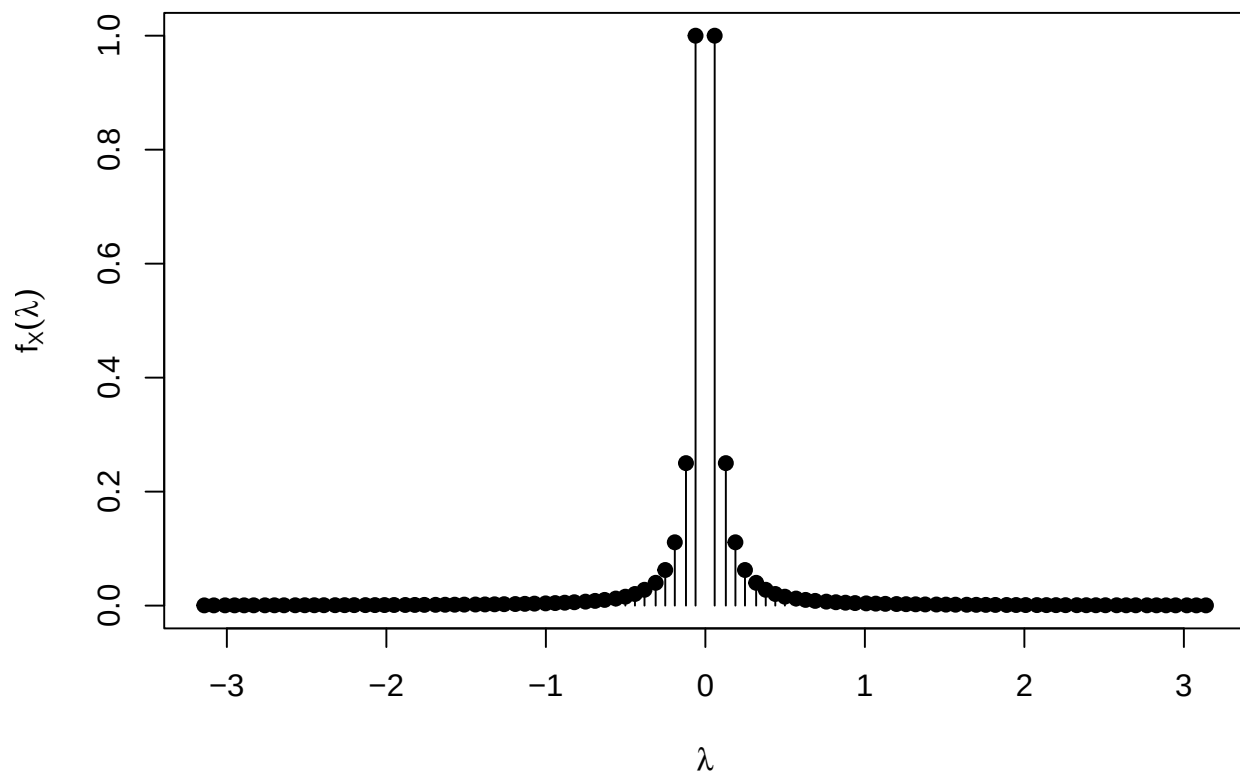
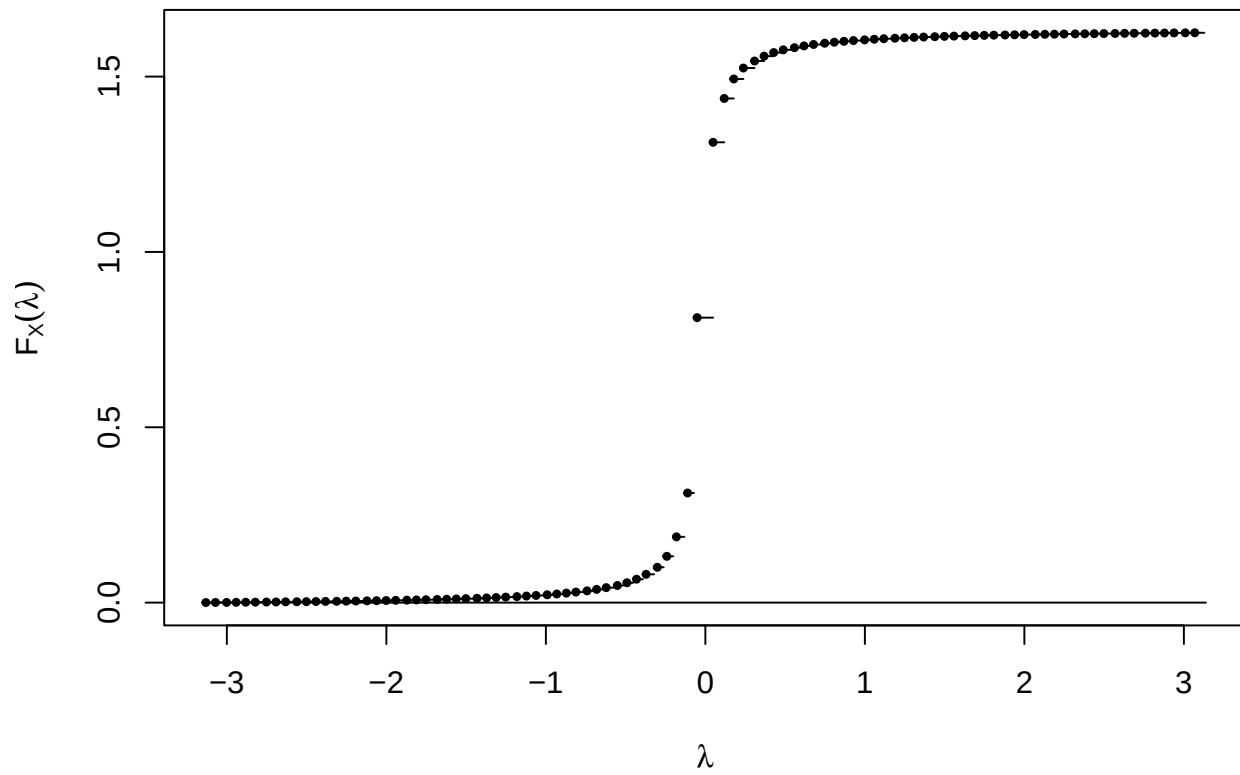
Finally suppose $k = 50$, with

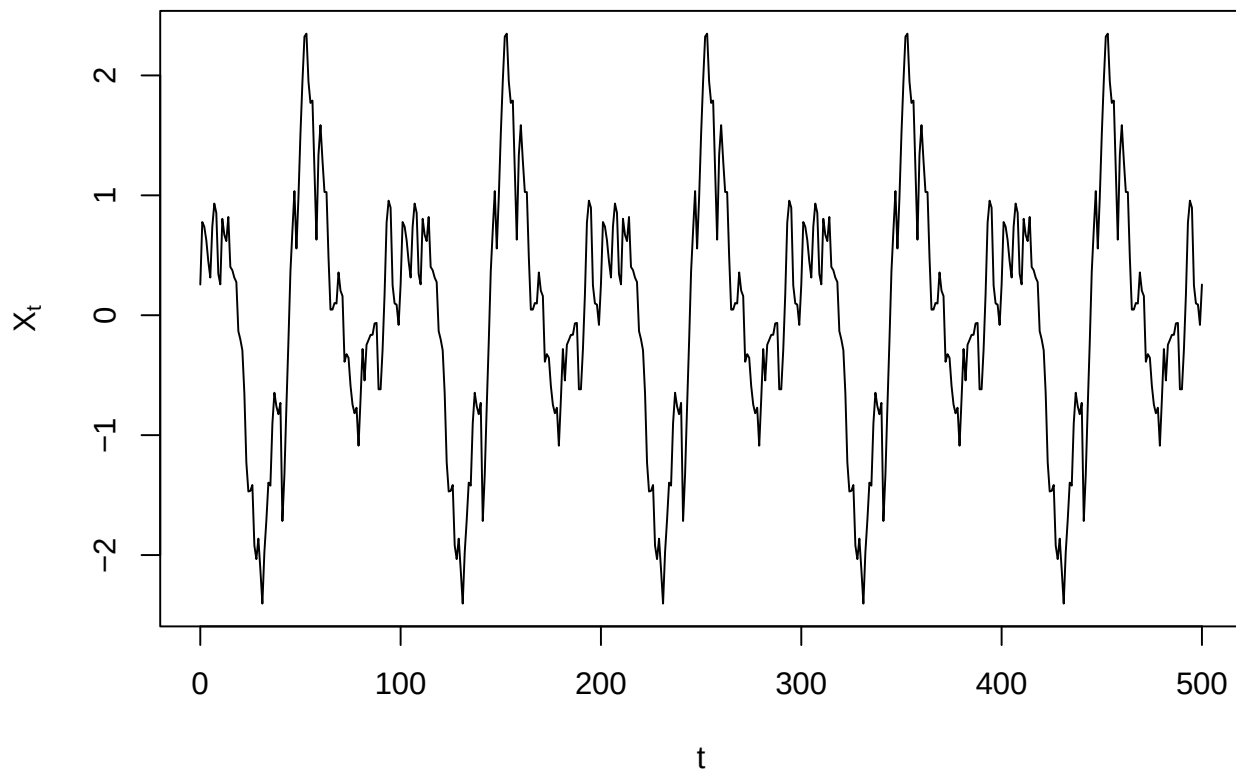
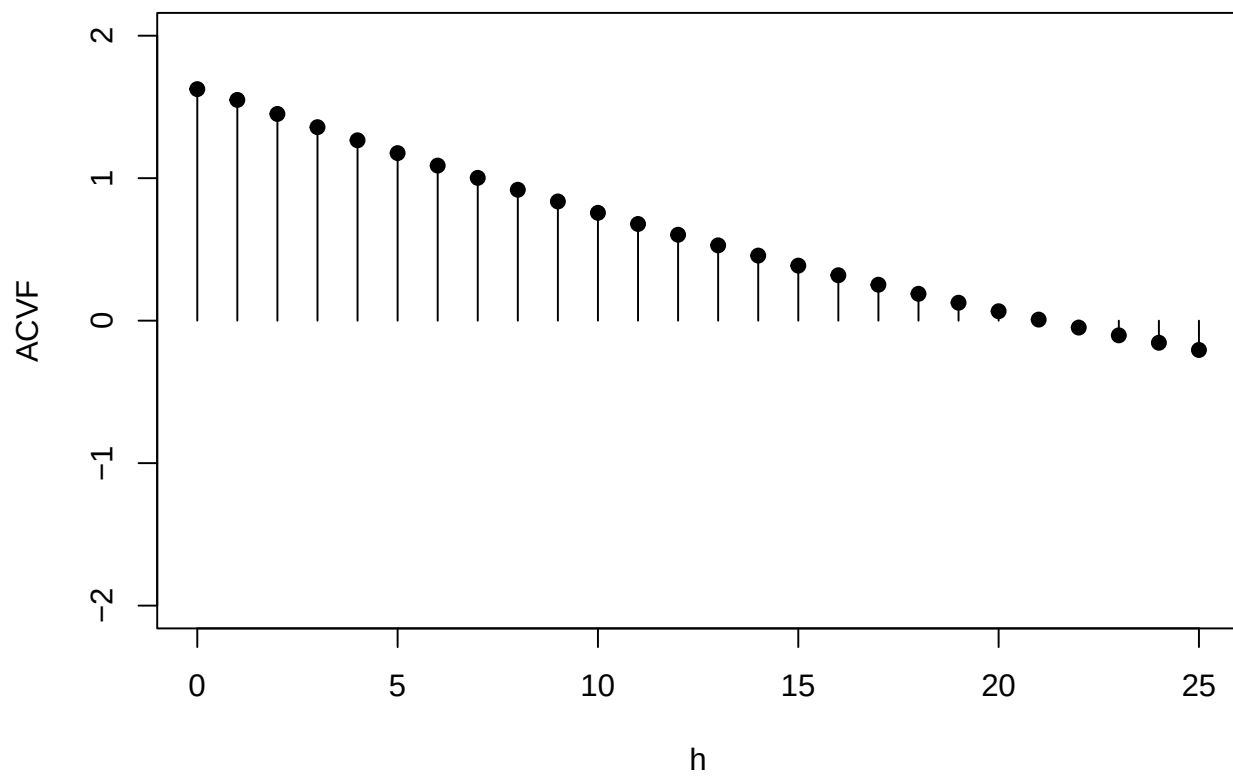
- $\omega_j = j\pi/k$,
- $\sigma_j^2 = 1/j^2$

for $j = 1, \dots, k$.

```
k<-50
lam<-seq(-pi,pi,by=0.01)
omega<-c(1:k)*pi/k
sigsq<-1/c(1:k)^2
hvec<-c(0:25)
gammah<-hvec*0
for(j in 1:k){
  gammah<-gammah+sigsq[j]*cos(omega[j]*hvec)
}
acvf<-cbind(hvec,gammah)
rownames(acvf)<-paste('h=',hvec,sep='')
colnames(acvf)<-c('h','ACVF')
acvf[1:10,2]
```

	h=0	h=1	h=2	h=3	h=4	h=5	h=6	h=7
##	1.6251327	1.5493617	1.4508704	1.3579329	1.2659031	1.1760842	1.0883780	1.0023049
	h=8	h=9						
##	0.9186677	0.8364641						





For these processes, the variance of $\{X_t\}$ can be decomposed into the variances contributed by the harmonic components at different frequencies $\omega_1, \dots, \omega_k$. In the limit as $k \rightarrow \infty$

$$X_t \stackrel{\text{m.s.}}{=} \int_{(-\pi, \pi]} e^{it\lambda} dZ(\lambda)$$

where $\{Z(\lambda) : -\pi < \lambda \leq \pi\}$ is a non-decreasing stochastic process. In general, Z is a *complex-valued* process which has uncorrelated increments, taking the form

$$X_t = X_{1,t} + iX_{2,t}$$

say, where $\mathbb{E}[X_t] = \mathbb{E}[X_{1,t}] + i\mathbb{E}[X_{2,t}]$ and

$$\mathbb{E}[X_t X_{t+h}] = \mathbb{E}[(X_{1,t} + iX_{2,t})(X_{1,t+h} + iX_{2,t+h})] = \mathbb{E}[X_{1,t}X_{1,t+h} - X_{2,t}X_{2,t+h}] + i\mathbb{E}[X_{1,t}X_{2,t+h} + X_{2,t}X_{1,t+h}]$$

etc. We have

$$X_t = \int_{(-\pi, \pi]} e^{it\lambda} dZ(\lambda) = \int_{(-\pi, \pi]} [\cos(t\lambda) + i\sin(t\lambda)] dZ(\lambda)$$

where $Z(\lambda)$ is a random process on $(-\pi, \pi]$. We need only consider $Z(\lambda)$ for $0 \leq \lambda \leq \pi$. For finite k , if

$$dZ(\lambda) = \begin{cases} \frac{U_j + iV_j}{2} & \lambda = -\omega_j, j = 1, \dots, k \\ \frac{U_j - iV_j}{2} & \lambda = \omega_j, j = 1, \dots, k \\ 0 & \text{otherwise} \end{cases}$$

Here

$$\mathbb{E}[dZ(\lambda) \overline{dZ(\lambda)}] = \mathbb{E}[(U_j^2 + V_j^2)/4] = \frac{\sigma_j^2}{2}$$

for $\lambda = \pm\omega_j, j = 1, \dots, k$ and zero otherwise. We have that

$$\begin{aligned} X_t &= \int_{(-\pi, \pi]} e^{it\lambda} dZ(\lambda) \\ &= \frac{1}{2} \sum_{j=1}^k (U_j + iV_j) e^{-i\omega_j t} + \frac{1}{2} \sum_{j=1}^k (U_j - iV_j) e^{i\omega_j t} \\ &= \sum_{j=1}^k [U_j \cos(\omega_j t) + V_j \sin(\omega_j t)] \end{aligned}$$

where, by the previous calculation, yields

$$\gamma_X(h) = \sum_{j=1}^k \sigma_j^2 \cos(\omega_j h)$$

Recall the earlier inversion formula

$$\gamma_X(h) = \int_{-\pi}^{\pi} \cos(h\lambda) f_X(\lambda) d\lambda = 2 \int_0^{\pi} \cos(h\lambda) f_X(\lambda) d\lambda.$$

This informs us that for a target $\gamma_X(h)$, we should choose the σ_j^2 quantities via the spectral density function. We can consider the limit of this process as $k \rightarrow \infty$; we require

$$\sum_{j=1}^{\infty} \sigma_j^2 < \infty$$

which implies (if the σ_j^2 values are sorted into descending order) that $\sigma_j^2 \rightarrow 0$ as $j \rightarrow \infty$.

Example: AR(1) process.

If $X_t \sim \text{AR}(1)$ with parameter ϕ , $-1 < \phi < 1$, we have that

$$\gamma_X(h) = \frac{\sigma_Z^2}{1 - \phi^2} \phi^h \quad h = 0, 1, 2, \dots$$

and

$$f_X(\lambda) = \frac{\sigma_Z^2}{2\pi} \frac{1}{|1 - \phi e^{-i\lambda}|^2} \quad -\pi < \lambda < \pi$$

This suggests that if we take $\omega_j = j\pi/k$, $j = 0, 1, \dots, k$ as a regular grid with spacing π/k , we should set

$$\sigma_j^2 = \frac{\sigma_Z^2}{k} \frac{1}{|1 - \phi e^{-i\omega_j}|^2}$$

as then, if k is large enough

$$\frac{\sigma_0^2}{2} + \sum_{j=1}^{k-1} \sigma_j^2 \cos(h\omega_j) + \frac{\sigma_k^2}{2} \cos(\pi h) \approx \int_{-\pi}^{\pi} \cos(h\lambda) f_X(\lambda) d\lambda$$

using a trapezium rule approximation.

```
k<-1000
phi<-0.8
sigZsq<-4
sigZsq/(1-phi^2)

## [1] 11.11111

lam<-seq(-pi,pi,by=0.01)
omega<-c(0:k)*pi/k
sigsq<-(sigZsq/k)*(1/Mod(1-phi*exp(-1i*omega))^2)
sum(sigsq[2:k])+0.5*(sigsq[1]+sigsq[k+1])

## [1] 11.11111

hvec<-c(0:25)
gammah<-0.5*sigsq[1]*cos(omega[1]*hvec)
for(j in 2:k){
  gammah<-gammah+sigsq[j]*cos(omega[j]*hvec)
}
gammah<-gammah+0.5*sigsq[k+1]*cos(omega[k+1]*hvec)

acvf<-cbind(hvec,gammah)
rownames(acvf)<-paste('h=',hvec,sep='')
colnames(acvf)<-c('h','ACVF')
acvf[1:10,2]

##          h=0          h=1          h=2          h=3          h=4          h=5          h=6          h=7
## 11.111111  8.888889  7.111111  5.688889  4.551111  3.640889  2.912711  2.330169
##          h=8          h=9
##  1.864135  1.491308

acvf[1:10,2]/acvf[1,2] #ACF

##          h=0          h=1          h=2          h=3          h=4          h=5          h=6          h=7
## 1.0000000  0.8000000  0.6400000  0.5120000  0.4096000  0.3276800  0.2621440  0.2097152
##          h=8          h=9
##  0.1677722  0.1342177
```

