

# MATH 545: INTRODUCTION TO TIME SERIES ANALYSIS

## WINTER 2025

### STATE SPACE AND DYNAMIC LINEAR MODELS

A *state space model* couples two possibly vector-valued time series processes,  $\{\mathbf{X}_t\}$  and  $\{\mathbf{Y}_t\}$  to extend the types of models that can be achieved via single processes. A general linear state space model (or *dynamic linear model*) is specified by two equations

$$\begin{array}{llll} \text{OBSERVATION EQUATION} & \mathbf{Y}_t & = & \mathbf{F}_t \mathbf{X}_t + \mathbf{d}_t + \mathbf{V}_t & K \times 1 \\ \text{STATE EQUATION} & \mathbf{X}_{t+1} & = & \mathbf{G}_t \mathbf{X}_t + \mathbf{e}_t + \mathbf{W}_t & L \times 1 \end{array}$$

where

- $\mathbf{F}_t$  is a  $K \times L$  deterministic matrix;
- $\mathbf{d}_t$  is a  $K \times 1$  deterministic vector;
- $\mathbf{G}_t$  is a  $L \times L$  deterministic matrix;
- $\mathbf{e}_t$  is a  $L \times 1$  deterministic vector.

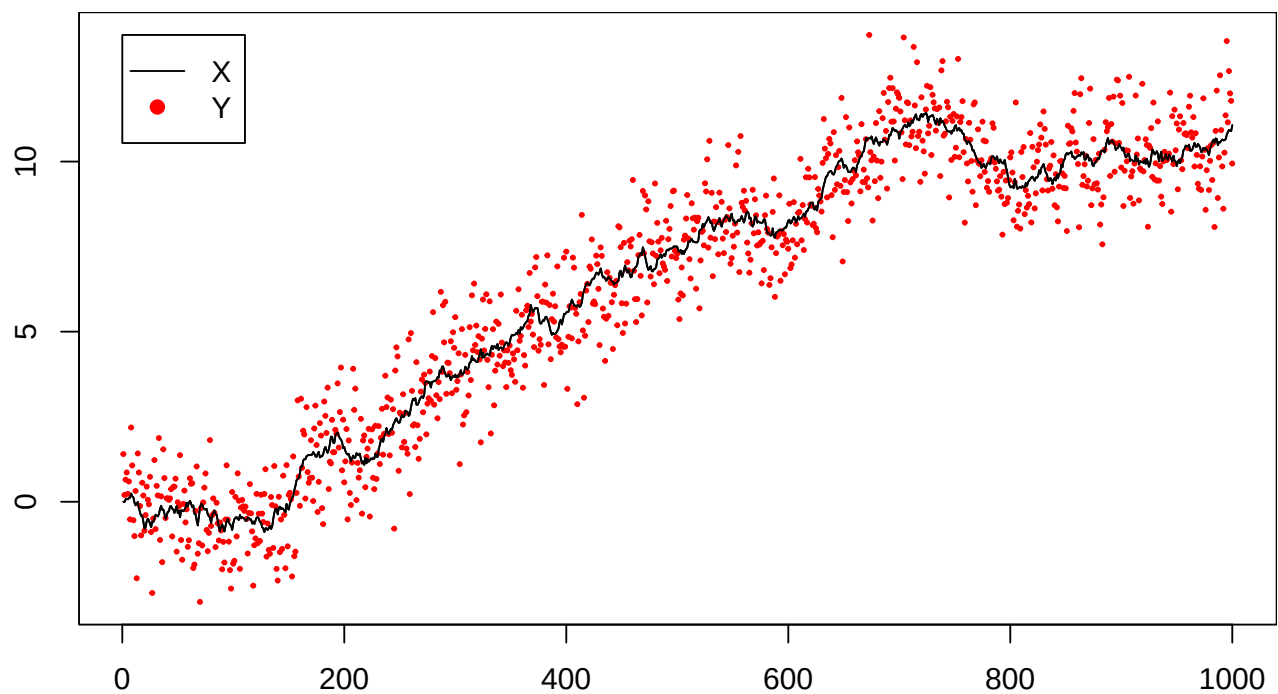
and the stochastic terms are  $\{\mathbf{V}_t\}$  and  $\{\mathbf{W}_t\}$  which are independent vector white noise processes. Thus  $\{\mathbf{X}_t\}$  is an  $L \times 1$  time series process that can be thought to represent unobserved (or latent) stochastic variation, and  $\{\mathbf{Y}_t\}$  is a  $K \times 1$  process that represents the observed process. To initialize, we also specify the initial state  $\mathbf{X}_1$ .

**EXAMPLE:** Suppose  $K = L = 1$ , and

$$\begin{array}{llll} \text{OBSERVATION EQUATION} & Y_t & = & X_t + V_t \\ \text{STATE EQUATION} & X_{t+1} & = & X_t + 0.01 + W_t \end{array}$$

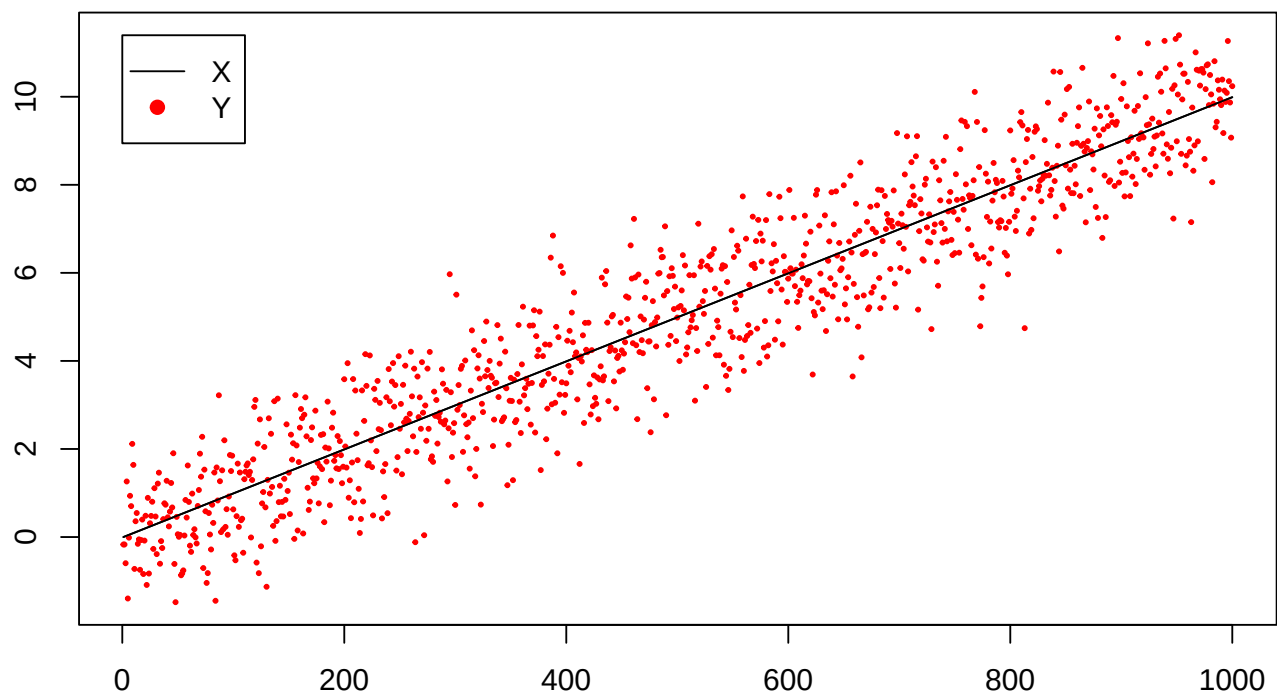
where  $V_t \sim \text{Normal}(0, \sigma_V^2)$  and  $W_t \sim \text{Normal}(0, \sigma_W^2)$ . We choose  $\sigma_V = 1$ ,  $\sigma_W = 0.1$  and  $X_1 = 0$ .

```
set.seed(3834)
n<-1000;X<-Y<-rep(0,n)
F<-G<-1;dvec<-0;evec<-0.01
sigW<-0.1;sigV<-1
W<-rnorm(n)*sigW
V<-rnorm(n)*sigV
X[1]<-0;Y[1]<-F*X[1]+dvec+V[1]
for(i in 2:n){
  X[i]<-G*X[i-1]+evec+W[i]
  Y[i]<-F*X[i]+dvec+V[i]
}
par(mar=c(3,3,0,0));plot(X,type='l',ylim=range(Y));points(Y,pch=19,cex=0.3,col='red');lines(X)
legend(0,max(Y),c('X','Y'),col=c('black','red'),pch=c(-1,19),lty=c(1,-1))
```



It is possible to form a **deterministic** latent structure by setting  $\sigma_W = 0$ .

```
F<-G<-1;dvec<-0;evec<-0.01; sigV<-1;sigW<-0.0
W<-rnorm(n)*sigW;V<-rnorm(n)*sigV
X[1]<-0; Y[1]<-F*X[1]+evec+V[1]
for(i in 2:n){
  X[i]<-G*X[i-1]+evec+W[i]
  Y[i]<-F*X[i]+dvec+V[i]
}
par(mar=c(2,3,1,0));plot(X,type='l',ylim=range(Y));points(Y,pch=19,cex=0.3,col='red');lines(X)
legend(0,max(Y),c('X','Y'),col=c('black','red'),pch=c(-1,19),lty=c(1,-1))
```



**EXAMPLE:** Suppose  $K = 1$  and  $L = 2$ , with  $\mathbf{X}_t = (X_{1,t}, X_{2(t-1)})^\top$ , and

$$\begin{aligned} \text{OBSERVATION EQUATION} \quad Y_t &= [1 \ 0] \mathbf{X}_t + V_t \\ \text{STATE EQUATION} \quad \mathbf{X}_{t+1} &= \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \mathbf{X}_t + \mathbf{W}_t \end{aligned}$$

where  $V_t \sim \text{Normal}(0, \sigma_V^2)$  and  $\mathbf{W}_t = (W_{1t}, W_{2t})^\top$ ,  $W_{1t} \sim \text{Normal}(0, \sigma_W^2)$ , and  $W_{2t} \equiv 0$  for all  $t$ , with  $\mathbb{E}[V_t W_{1s}] = 0$  for all  $t, s$ . Note that this model specifies an AR(2) model for  $\{X_{1,t}\}$ , the first component of  $\{\mathbf{X}_t\}$ , as

$$X_{1,(t+1)} = \phi_1 X_{1,t} + \phi_2 X_{1,t-1} + W_{1t}.$$

with  $Y_t = X_{1,t} + V_t$ .

In the following simulation, we choose  $\phi_1 = 1.7$ ,  $\phi_2 = -0.72$ ,  $\sigma_W = 1$ ,  $\sigma_V = 0.5$  and  $\mathbf{X}_1 = (0, 0)^\top$ . The process for  $\mathbf{X}_{1,t}$  is stationary, but highly autocorrelated as the AR(2) polynomial is

$$\Phi(z) = 1 - 1.7z + 0.72z^2 = (1 - 0.8z)(1 - 0.9z)$$

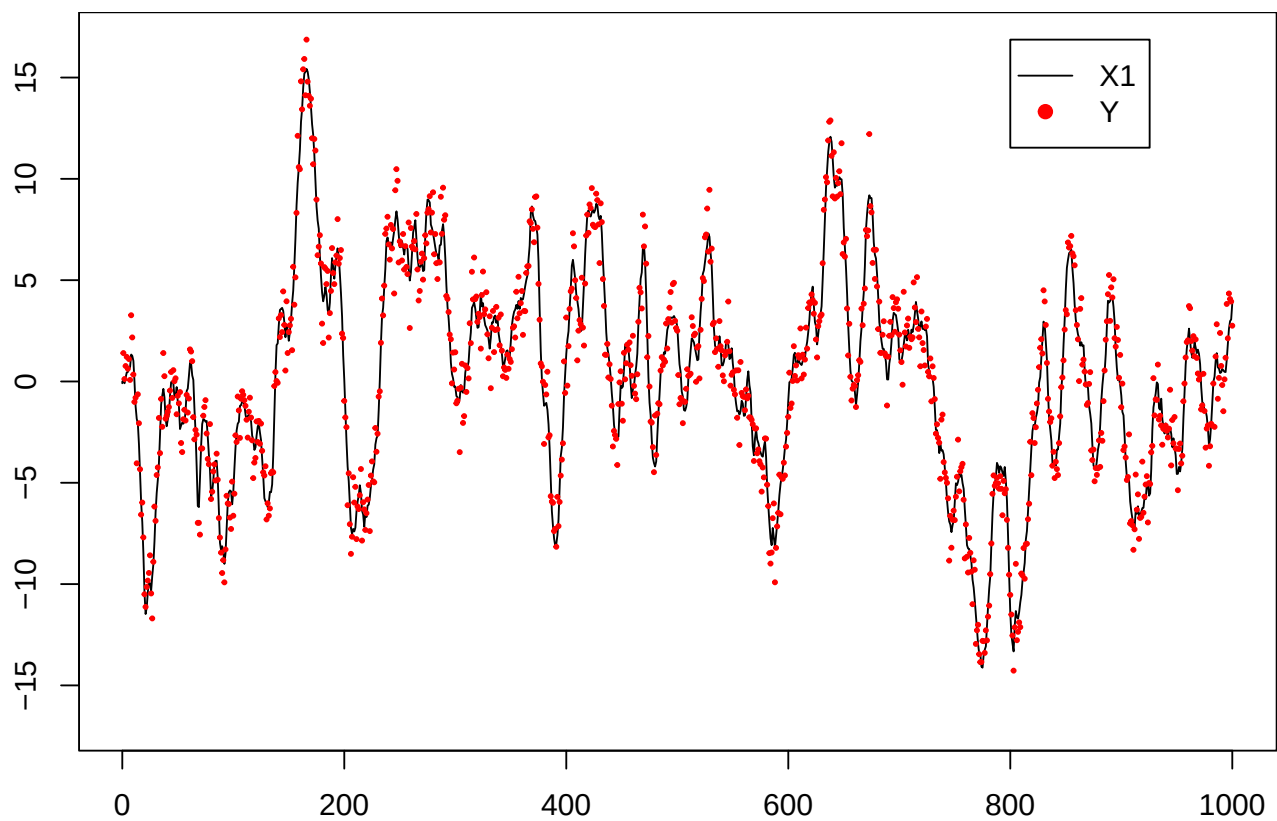
so both reciprocal roots are close to one. In this case,  $\{Y_t\}$  has zero mean, and  $\text{Var}[Y_t] = \gamma_X(0) + \sigma_V^2$ . Then

$$\begin{aligned} \mathbb{E}[Y_t Y_{t+h}] &= \mathbb{E}[(X_{1,t} + V_t)(X_{1,t+h} + V_{t+h})] \\ &= \mathbb{E}[X_{1,t} X_{1,t+h}] + \mathbb{E}[X_{1,t} V_{t+h}] + \mathbb{E}[V_t X_{1,t+h}] + \mathbb{E}[V_t V_{t+h}] \\ &= \gamma_X(h) \end{aligned}$$

as  $\mathbb{E}[V_t] = 0$  for all  $t$ , and  $\mathbb{E}[X_{1,t} V_s] = \mathbb{E}[X_{1,t}] \mathbb{E}[V_s] = 0$  for all  $t$  and  $s$ . Therefore the resulting process  $\{Y_t\}$  is **stationary**, with autocovariance function

$$\gamma_Y(h) \equiv \gamma_X(h) \quad |h| \geq 1.$$

```
set.seed(3834)
n<-1000;X<-matrix(0,ncol=n,nrow=2)
Y<-rep(0,n)
F<-c(1,0)
phi1<-1.7;phi2<-0.72;G<-matrix(c(phi1,1,phi2,0),2,2)
sigV<-1;sigW<-0.5
W1<-rnorm(n)*sigW
V<-rnorm(n)*sigV
Y[1]<-(F%*%X[,1])[1,1]+V[1]
for(i in 2:n){
  X[,i]<-G%*%X[,i-1]+c(W1[i],0)
  Y[i]<-(F%*%X[,i])[1,1]+V[i]
}
par(mar=c(2,3,0,0));plot(X[1,],type='l',ylim=range(-max(abs(Y)),max(abs(Y))))
points(Y,pch=19,cex=0.3,col='red');lines(X)
legend(800,max(abs(Y)),c('X1','Y'),col=c('black','red'),pch=c(-1,19),lty=c(1,-1))
```



```

set.seed(3834)
n<-100000
X<-matrix(0,ncol=n,nrow=2)
Y<-rep(0,n)
F<-c(1,0)
phi1<-1.7;phi2<--0.72
G<-matrix(c(phi1,1,phi2,0),2,2)
sigW<-1;sigV<-0.5
W1<-rnorm(n)*sigW
V<-rnorm(n)*sigV
Y[1]<-(F%*%X[,1])[1,1]+V[1]
for(i in 2:n){
  X[,i]<-G%*%X[,i-1]+c(W1[i],0)
  Y[i]<-(F%*%X[,i])[1,1]+V[i]
}
psi1<-ARMAtoMA(ar=c(phi1,phi2),lag.max=50)
gammaX0<-sigV^2*(1+sum(psi1^2))
true.acv.Y<-gammaX0*ARMAacf(ar=c(phi1,phi2),lag.max=50)
true.acv.Y[1]<-true.acv.Y[1]+sigW^2; true.acf.Y<-true.acv.Y/true.acv.Y[1]
acf.Y<-acf(Y,lag.max=50,plot=FALSE)$acf
rbind(acf.Y[1:7],true.acf.Y[1:7])

+      0      1      2      3      4      5      6
+ [1,] 1 0.9855056 0.9571886 0.9175362 0.8704643 0.8190079 0.7654673
+ [2,] 1 0.9462234 0.9192839 0.8815018 0.8366686 0.7876553 0.7366127

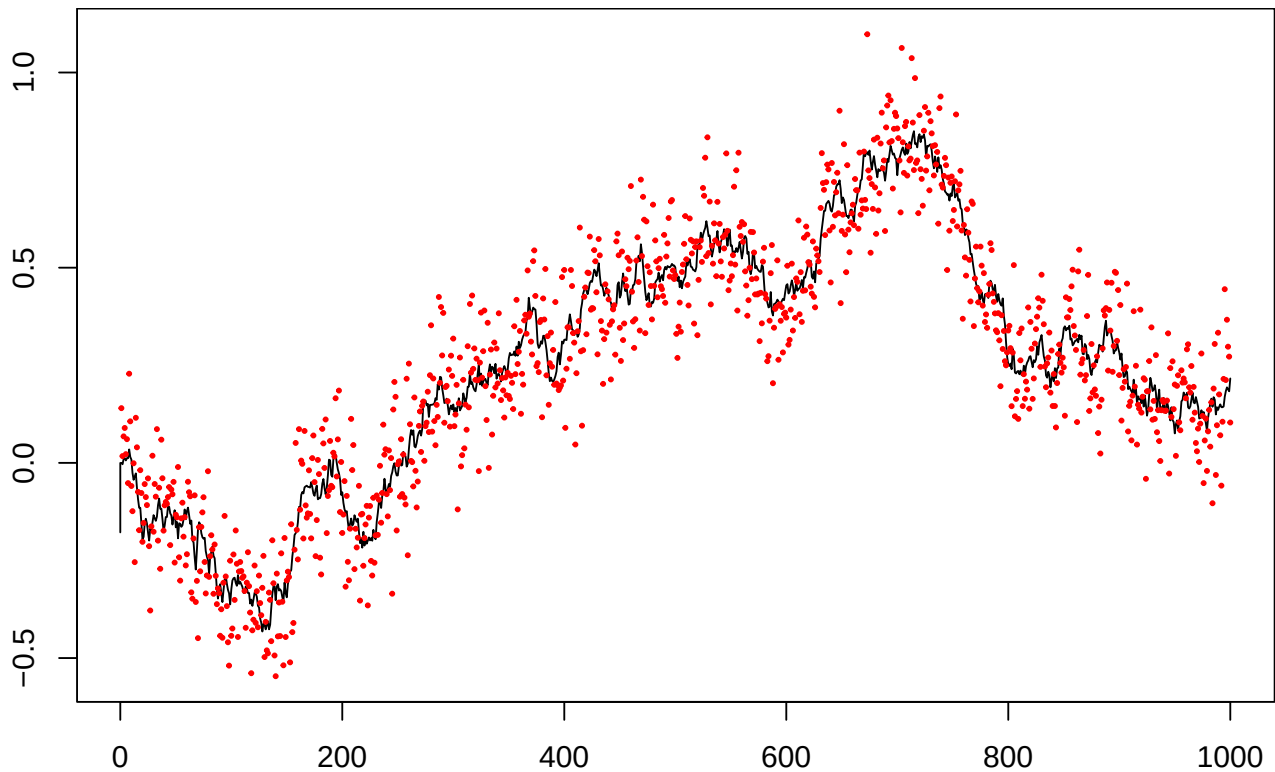
```

**EXAMPLE: LOCAL LEVEL MODEL**  $K = L = 1$ , and

$$\begin{array}{ll} \text{OBSERVATION EQUATION} & Y_t = M_t + V_t \\ \text{STATE EQUATION} & M_{t+1} = M_t + W_t \end{array}$$

where  $V_t \sim \text{Normal}(0, \sigma_V^2)$  and  $W_t \sim \text{Normal}(0, \sigma_W^2)$ . In the following simulation, we choose  $\sigma_V = 0.1, \sigma_W = 0.02$  and  $M_1 = 0$ .

```
set.seed(3834)
n<-1000;M<-Y<-rep(0,n)
G<-F<-1
sigW<-0.02;sigV<-0.1
W<-rnorm(n)*sigW
V<-rnorm(n)*sigV
M[1]<-0;Y[1]<-F*X[1]+V[1]
for(i in 2:n){
  M[i]<-G*M[i-1]+W[i]
  Y[i]<-F*M[i]+V[i]
}
par(mar=c(2,3,0,0));plot(M,type='l',ylim=range(Y));points(Y,pch=19,cex=0.3,col='red');lines(X)
legend(800,2,c('X','Y'),col=c('black','red'),pch=c(-1,19),lty=c(1,-1))
```



Here, the process  $\{Y_t\}$  is **non-stationary**: we clearly have that

$$\text{Var}[Y_t] = \text{Var}[M_t] + \text{Var}[W_t] = \text{Var}[M_t] + \sigma_W^2$$

but  $\{M_t\}$  is a random walk, so it has a variance that is dependent on  $t$

$$\text{Var}[M_t] = t\sigma_V^2.$$

**EXAMPLE: LOCAL LINEAR MODEL**  $K = 1, L = 2$  and

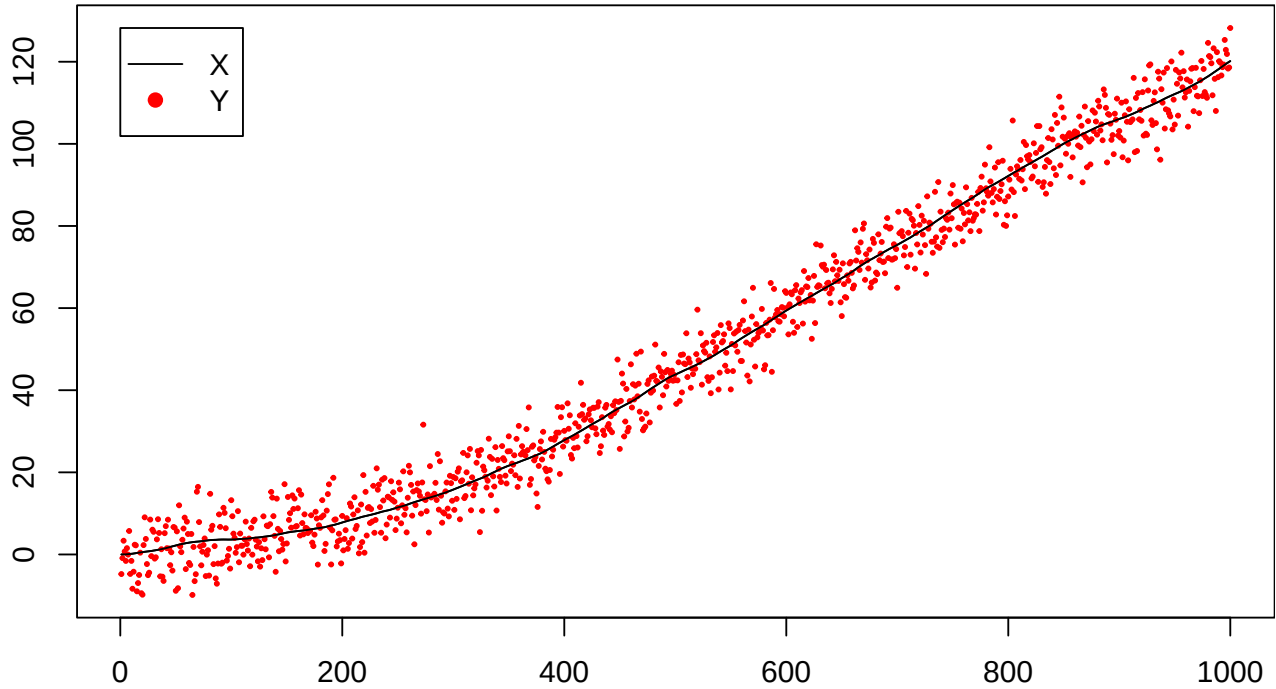
$$\begin{array}{ll} \text{OBSERVATION EQUATION} & Y_t = M_t + V_t \\ \text{STATE EQUATION} & \begin{aligned} M_{t+1} &= M_t + B_t + W_{1t} \\ B_{t+1} &= B_t + W_{2t} \end{aligned} \end{array}$$

where we define the state  $\mathbf{X}_t = (M_t, B_t)^\top$ , and  $\mathbf{W}_t = (W_{1t}, W_{2t})^\top$ , so that

$$\mathbf{G}_t = \mathbf{G}_L = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \mathbf{F}_t = [1 \ 0].$$

We have  $V_t \sim \text{Normal}(0, \sigma_V^2)$  and  $W_{1t} \sim \text{Normal}(0, \sigma_{W_1}^2)$ ,  $W_{2t} \sim \text{Normal}(0, \sigma_{W_2}^2)$  with  $W_{1t}$  and  $W_{2t}$  independent. In the following simulation, we choose  $\sigma_V = 5$ ,  $\sigma_{W_1} = 0.001$  and  $\sigma_{W_2} = 0.005$ , with  $M_1 = B_1 = 0$ .

```
set.seed(3834)
n<-1000;X<-matrix(0,ncol=n,nrow=2);Y<-rep(0,n)
F<-c(1,0)
G<-matrix(c(1,0,1,1),2,2)
sigV<-5;sigW1<-0.001;sigW2<-0.005
V<-rnorm(n)*sigV;W1<-rnorm(n)*sigW1;W2<-rnorm(n)*sigW2
X[,1]<-0;
Y[1]<-(F%*%X[,1])[1,1]+V[1]
for(i in 2:n){
  X[,i]<-G%*%X[,i-1]+c(W1[i],W2[i])
  Y[i]<-(F%*%X[,i])[1,1]+V[i]
}
mt<-apply(X,2,sum)
par(mar=c(2,3,1,0));plot(mt,type='l',ylim=range(Y));points(Y,pch=19,cex=0.3,col='red')
lines(mt);legend(0,max(Y),c('X','Y'),col=c('black','red'),pch=c(-1,19),lty=c(1,-1))
```



This model is suitable for carrying out non-linear regression; the smoothness of the underlying function as modelled by the states is determined by the magnitude of  $\sigma_{W_1}$  and  $\sigma_{W_2}$ .

**EXAMPLE: SEASONAL MODEL** In a deterministic purely seasonal model, with seasonality  $r$  lags, we have  $Y_t = s_t + V_t$ , say, where  $s_t = s_{t-r}$  for all  $t$ , such that  $\sum_{t=1}^r s_t = 0$ . That is,

$$s_{t+1} = -s_t - s_{t-1} - \cdots - s_{t-r+2}.$$

To construct a stochastic seasonal model, we may write

$$S_{t+1} = -S_t - S_{t-1} - \cdots - S_{t-r+2} + W_{1t}$$

where  $\{W_{1t}\} \sim \text{WN}(0, \sigma_S^2)$ , say. We write this as a state space model as follows: we have  $K = 1$ ,  $L = (r - 1)$  and

$$\begin{array}{ll} \text{OBSERVATION EQUATION} & Y_t = \mathbf{F}_t \mathbf{X}_t + V_t \\ \text{STATE EQUATION} & \mathbf{X}_{t+1} = \mathbf{G}_t \mathbf{X}_t + \mathbf{W}_t \end{array}$$

where we define the state  $\mathbf{X}_t = (Y_t, Y_{t-1}, \dots, Y_{t-r+2})^\top$ , and  $\mathbf{W}_t = (S_t, 0, 0, \dots, 0)^\top$ , so that

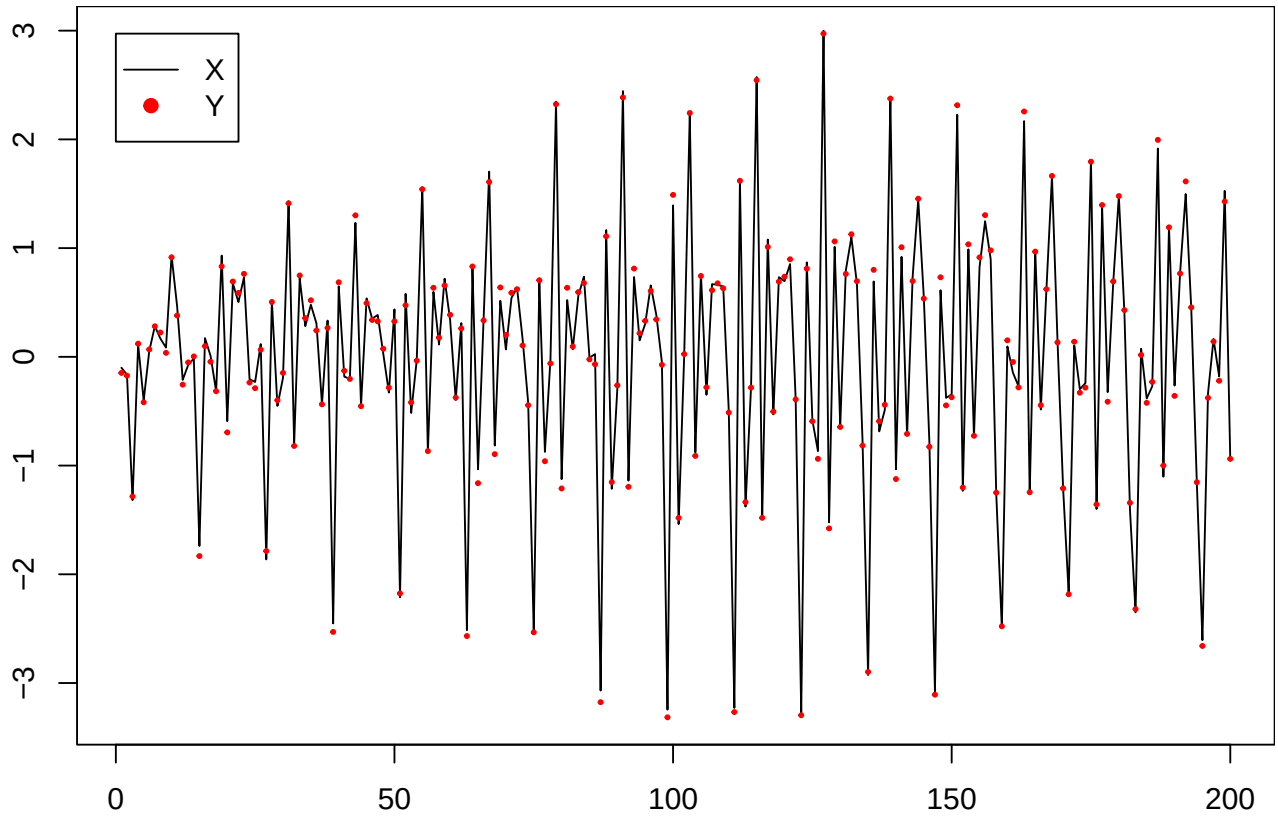
$$\mathbf{G}_t \equiv \mathbf{G}_S = \begin{bmatrix} -1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

and

$$\mathbf{F}_t = [1 \quad 0 \quad 0 \quad \cdots \quad 0]$$

with  $V_t \sim \text{Normal}(0, \sigma_V^2)$  and  $S_t \sim \text{Normal}(0, \sigma_S^2)$ . In the following simulation, with  $r = 12$ , we choose  $\sigma_V = 0.05$ ,  $\sigma_S = 0.2$ , and with  $\mathbf{X}_t = (-0.1, 0.1, 0.3, 0.7, 0.4, 0.2, -0.1, 0.0, -0.1, 0.0, -1.4)^\top$ .

```
set.seed(3834); n<-200; r<-12
X<-matrix(0, ncol=n, nrow=r-1); Y<-rep(0, n)
F<-c(1, rep(0, r-2))
G<-matrix(0, r-1, r-1)
G[1,]<-rep(-1, r-1); for(j in 2:(r-1)){G[j, j-1]<-1}
sigV<-0.05; sigS<-0.2
V<-rnorm(n)*sigV; S<-rnorm(n)*sigS
X[,1]<-c(-0.1, 0.1, 0.3, 0.7, 0.4, 0.2, -0.1, 0.0, -0.1, 0.0, -1.4)
Y[1]<-(F%*%X[,1])[1,1]+V[1]
for(i in 2:n){
  Wi<-c(S[i], rep(0, r-2))
  X[,i]<-G%*%X[,i-1]+Wi
  Y[i]<-(F%*%X[,i])[1,1]+V[i]
}
par(mar=c(2,3,0,0)); plot(X[1,], type='l', ylim=range(Y)); points(Y, pch=19, cex=0.3, col='red')
legend(0, max(Y), c('X', 'Y'), col=c('black', 'red'), pch=c(-1, 19), lty=c(1, -1))
```



**EXAMPLE: LOCAL LINEAR + SEASONAL MODEL** We can combine the two previous models to construct a state space model as follows: we have  $K = 1$ ,  $L = (r + 1)$  and

$$\begin{array}{ll} \text{OBSERVATION EQUATION} & Y_t = \mathbf{F}_t \mathbf{X}_t + V_t \\ \text{STATE EQUATION} & \mathbf{X}_{t+1} = \mathbf{G}_t \mathbf{X}_t + \mathbf{W}_t \end{array}$$

where we define the state  $\mathbf{X}_t = (M_t, B_t, S_t, S_{t-1}, \dots, S_{t-r+2})^\top$ , and  $\mathbf{W}_t = (W_{1t}, W_{2t}, W_{3t}, 0, \dots, 0)^\top$ , so that

$$\mathbf{G}_t \equiv \mathbf{G} = \begin{bmatrix} \mathbf{G}_L & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_S \end{bmatrix}$$

and

$$\mathbf{F}_t = [1 \quad 0 \quad 1 \quad 0 \quad \dots \quad 0]$$

with  $V_t \sim \text{Normal}(0, \sigma_V^2)$  and  $W_{3t} \sim \text{Normal}(0, \sigma_S^2)$ . In the following simulation, we choose  $\sigma_V = 1$ ,  $\sigma_{W_1} = 0.1$ ,  $\sigma_{W_2} = 0.05$ , and  $\sigma_S = 0.5$ , with  $M_1 = B_1 = 0$ , and with  $\mathbf{X}_1 = (-0.1, 0.1, 0.3, 0.7, 0.4, 0.2, -0.1, 0.0, -0.1, 0.0, -1.4)^\top$ .

```
set.seed(3834)
n<-200;r<-12
X<-matrix(0,ncol=n,nrow=r+1);Y<-mt<-rep(0,n)
F<-c(1,0,1,rep(0,r-2))
G<-matrix(0,r+1,r+1)
G[1:2,1:2]<-matrix(c(1,0,1,1),2,2)
G[3,]<-c(0,0,rep(-1,r-1));for(j in 4:(r+1)){G[j,j-1]<-1}
sigV<-1
sigW1<-0.1
sigW2<-0.05
sigS<-0.5
V<-rnorm(n)*sigV;W1<-rnorm(n)*sigW1;W2<-rnorm(n)*sigW2;S<-rnorm(n)*sigS
X[,1]<-c(0,0,-0.1,0.1,0.3,0.7,0.4,0.2,-0.1,0.0,-0.1,0.0,-1.4)
mt[1]<-(F%*%X[,1])[1,1]
Y[1]<-mt[1]+V[1]
```

```

for(i in 2:n){
  Wi<-c(W1[i],W2[i],S[i],rep(0,r-2))
  X[,i]<-G%*%X[,i-1]+Wi
  mt[i]<-(F%*%X[,i])[1,1]
  Y[i]<-mt[i]+V[i]
}
par(mar=c(2,3,0,0))
plot(mt,type='l',ylim=range(Y));points(Y,pch=19,cex=0.3,col='red')
lines(X[1,],col='blue')
legend(0,max(Y),c(expression(X[t]),expression(M[t]),expression(Y[t])),
  col=c('black','blue','red'), pch=c(-1,-1,19),cex=c(1,1,1),lty=c(1,1,-1))

```

