

MATH 545: INTRODUCTION TO TIME SERIES ANALYSIS

WINTER 2025

VECTOR AUTOREGRESSION MODELS: STATIONARITY

For the VAR(1) model

$$\mathbf{X}_t = \phi_0 + \Phi_1 \mathbf{X}_{t-1} + \mathbf{Z}_t$$

that, taking expectations, from stationarity

$$\boldsymbol{\mu} = \phi_0 + \Phi_1 \boldsymbol{\mu} \quad \therefore \quad \boldsymbol{\mu} = (\mathbf{I}_K - \Phi_1)^{-1} \phi_0$$

provided $(\mathbf{I}_K - \Phi_1)$ is non-singular. Then

$$\mathbf{X}_t - \boldsymbol{\mu} = \Phi_1(\mathbf{X}_{t-1} - \boldsymbol{\mu}) + \mathbf{Z}_t$$

If $\mathbf{Y}_t = \mathbf{X}_t - \boldsymbol{\mu}$, then by recursion

$$\mathbf{Y}_t = \mathbf{Z}_t + \Phi_1 \mathbf{Z}_{t-1} + \Phi_1^2 \mathbf{Z}_{t-2} + \cdots$$

and hence $\text{Cov}[\mathbf{Z}_t, \mathbf{Y}_{t-h}] = \text{Cov}[\mathbf{Z}_t, \mathbf{X}_{t-h}] = \mathbf{0}$, for all $h > 0$. In addition,

$$\mathbf{Y}_t \mathbf{Z}_t^\top = \mathbf{Z}_t \mathbf{Z}_t^\top + \Phi_1 \mathbf{Z}_{t-1} \mathbf{Z}_t^\top + \Phi_1^2 \mathbf{Z}_{t-2} \mathbf{Z}_t^\top + \cdots$$

and taking expectations yields

$$\text{Cov}[\mathbf{Y}_t, \mathbf{Z}_t] = \mathbb{E}[\mathbf{Y}_t \mathbf{Z}_t^\top] = \mathbb{E}[\mathbf{Z}_t \mathbf{Z}_t^\top] = \Sigma$$

as $\{\mathbf{Z}_t\}$ is a white-noise process. In the representation, $\mathbf{X}_t$ depends on $\mathbf{Z}_{t-j}$ with coefficient matrix Φ_1^j ; this dependence must converge to zero. Therefore, using the *eigendecomposition*

$$\Phi_1 = \mathbf{A}_1 \Lambda_1 \mathbf{B}_1$$

with $\mathbf{A}_1 \mathbf{B}_1 = \mathbf{I}_K$ and $\Lambda_1 = \text{diag}(\lambda_{11}, \dots, \lambda_{1K})$, we have in the usual way

$$\Phi_1^j = \mathbf{A}_1 \Lambda_1^j \mathbf{B}_1.$$

Thus, for each j ,

$$\Phi_1^j \longrightarrow \mathbf{0} \quad \Longleftrightarrow \quad |\lambda_{1k}| < 1, \quad k = 1, \dots, K$$

as in the univariate case. Note that

$$\det(\lambda \mathbf{I}_K - \Phi_1) = \lambda^K \det(\mathbf{I}_K - \lambda^{-1} \Phi_1)$$

and hence $\lambda_{11}, \dots, \lambda_{1K}$ are found via the solutions of

$$\det(\mathbf{I}_K - \Phi_1 z) = 0.$$

Similar to the univariate case, $\mathbf{X}_t$ is stationary if and only if the roots of

$$\det(z \mathbf{I}_K - \Phi_1) = 0$$

are greater than 1 in modulus. Finally

$$\Gamma_0 = \text{Var}[\mathbf{X}_t] = \text{Var}[\mathbf{Z}_t] + \sum_{j=1}^{\infty} \text{Var}[\Phi_1^j \mathbf{Z}_{t-j}] = \Sigma + \sum_{j=1}^{\infty} \Phi_1^j \Sigma (\Phi_1^\top)^j$$

Using the equation

$$\mathbf{X}_t - \boldsymbol{\mu} = \Phi_1(\mathbf{X}_{t-1} - \boldsymbol{\mu}) + \mathbf{Z}_t$$

we have for $h > 0$,

$$\mathbb{E}[(\mathbf{X}_t - \boldsymbol{\mu})(\mathbf{X}_{t-h} - \boldsymbol{\mu})^\top] = \Phi_1 \mathbb{E}[(\mathbf{X}_{t-1} - \boldsymbol{\mu})(\mathbf{X}_{t-h} - \boldsymbol{\mu})^\top] + \mathbb{E}[\mathbf{Z}_t(\mathbf{X}_{t-h} - \boldsymbol{\mu})^\top]$$

or equivalently, as the final expectation is zero

$$\mathbf{\Gamma}_h = \mathbf{\Phi}_1 \mathbf{\Gamma}_{h-1} = \mathbf{\Phi}_1^h \mathbf{\Gamma}_0$$

by recursion. Thus

$$\begin{aligned} \boldsymbol{\rho}_h &= \mathbf{D}^{-1/2} \mathbf{\Phi}_1 \mathbf{\Gamma}_{h-1} \mathbf{D}^{-1/2} \\ &= \left\{ \mathbf{D}^{-1/2} \mathbf{\Phi}_1 \mathbf{D}^{1/2} \right\} \left\{ \mathbf{D}^{-1/2} \mathbf{\Gamma}_{h-1} \mathbf{D}^{-1/2} \right\} \\ &= \mathbf{\Upsilon}_1 \boldsymbol{\rho}_{h-1} \\ &\vdots \\ &= \mathbf{\Upsilon}_1^h \boldsymbol{\rho}_0 \end{aligned}$$

say. Recall that

$$[\mathbf{\Gamma}_h]_{ij} = \mathbb{E}[(X_{i,t} - \mu_i)(X_{j,t-h} - \mu_j)] = \Gamma_{ij}(h)$$

Example: Let

$$\begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix} + \begin{bmatrix} 0.2 & 0.3 \\ -0.6 & 1.1 \end{bmatrix} \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \end{bmatrix} + \begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix}$$

with

$$\Sigma = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Then

$$\boldsymbol{\mu} = (\mathbf{I}_K - \mathbf{\Phi}_1)^{-1} \boldsymbol{\phi}_0 = \begin{bmatrix} 1 - 0.2 & -0.3 \\ 0.6 & 1 - 1.1 \end{bmatrix}^{-1} \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

```
library(mvnfast)
n<-10000
Sigma<-matrix(c(2,1,1,1),nrow=2,byrow=T)
Phi<-matrix(c(0.2,0.3,-0.6,1.1),nrow=2,byrow=T)
Phi0<-c(0.2,0.4)
Zt<-rmvn(n,c(0,0),Sigma)
Xt<-matrix(0,nrow=n,ncol=2)
for(i in 2:n){
  Xt[i,]<-Phi0+Phi %*% Xt[i-1,] + Zt[i,]
}
apply(Xt,2,mean)

## [1] 1.018430 2.044426

muvec<-solve(diag(c(1,1))-Phi) %*% Phi0
muvec

##      [,1]
## [1,]    1
## [2,]    2
```

For the eigendecomposition

```
eigen(Phi)

## eigen() decomposition
## $values
## [1] 0.8 0.5
##
## $vectors
```

```
##           [,1]      [,2]
## [1,] -0.4472136 -0.7071068
## [2,] -0.8944272 -0.7071068

lambdas<-as.numeric(eigen(Phi)$values)
Lambda1<-diag(diag(lambdas))
A1<-eigen(Phi)$vectors
B1<-solve(A1)
A1 %*% diag(Lambda1) %*% B1

##           [,1] [,2]
## [1,] 0.2 0.3
## [2,] -0.6 1.1
```

Thus we have

$$\mathbf{A}_1 = \begin{bmatrix} -0.4472136 & -0.7071068 \\ -0.8944272 & -0.7071068 \end{bmatrix} \quad \mathbf{B}_1 = \begin{bmatrix} 2.236068 & -2.236068 \\ -2.8284271 & 1.4142136 \end{bmatrix}$$

and

$$\mathbf{\Lambda}_1 = \begin{bmatrix} 0.8 & 0.0 \\ 0.0 & 0.5 \end{bmatrix}$$

We check that

$$\det(\mathbf{I}_K - \lambda^{-1} \mathbf{\Phi}_1) = 0$$

for $\lambda_{1,1} = 0.8$ and $\lambda_{1,2} = 0.5$.

```
det(diag(c(1,1))-(1/lambdas[1])*Phi)
## [1] 0.00000000000000004163336
det(diag(c(1,1))-(1/lambdas[2])*Phi)
## [1] 0
```

We can compute $\mathbf{\Gamma}_0$ using the formula

$$\mathbf{\Gamma}_0 = \mathbf{\Sigma} + \sum_{j=1}^{\infty} \mathbf{\Phi}_1^j \mathbf{\Sigma} (\mathbf{\Phi}_1^T)^j$$

and then $\mathbf{\Gamma}_h = \mathbf{\Phi}_1^h \mathbf{\Gamma}_0$.

```
Gamma0<-Sigma
Gammah<-array(0,c(200,2,2))
Rhoh<-array(0,c(200,2,2))
Phij<-Phi
PhijT<-t(Phi)
for(j in 1:200){
  Gamma0<-Gamma0+Phij %*% (Sigma %*% PhijT)
  Phij<-Phij %*% Phi
  PhijT<-PhijT %*% t(Phi)
}
Gamma0

##           [,1]      [,2]
## [1,] 2.777778 2.222222
## [2,] 2.222222 4.444444

var(Xt)

##           [,1]      [,2]
## [1,] 2.725313 2.184926
## [2,] 2.184926 4.408397
```

The theoretical and sample versions are similar. For the autocovariance matrices

```
Dmat<-diag(sqrt(diag(Gamma0)))
Phi<-Phi
for(j in 1:200){
  Gammah[j,,]<-Phi %*% Gamma0
  Rhoh[j,,]<-solve(Dmat) %*% (Gammah[j,,] %*% solve(Dmat))
  Phi<-Phi %*% Phi
}
```

```
Gammah[1,,] #h=1
```

```
##           [,1]      [,2]
## [1,]  1.2222222  1.777778
## [2,]  0.7777778  3.555556
```

```
Gammah[2,,] #h=2
```

```
##           [,1]      [,2]
## [1,]  0.4777778  1.422222
## [2,]  0.1222222  2.844444
```

```
Gammah[3,,] #h=3
```

```
##           [,1]      [,2]
## [1,]  0.1322222  1.137778
## [2,] -0.1522222  2.275556
```

```
Gammah[4,,] #h=4
```

```
##           [,1]      [,2]
## [1,] -0.01922222  0.9102222
## [2,] -0.24677778  1.8204444
```

and for the autocorrelation matrices

```
Rhoh[1,,] #h=1
```

```
##           [,1]      [,2]
## [1,]  0.4400000  0.5059644
## [2,]  0.2213594  0.8000000
```

```
Rhoh[2,,] #h=2
```

```
##           [,1]      [,2]
## [1,]  0.1720000  0.4047715
## [2,]  0.03478505  0.6400000
```

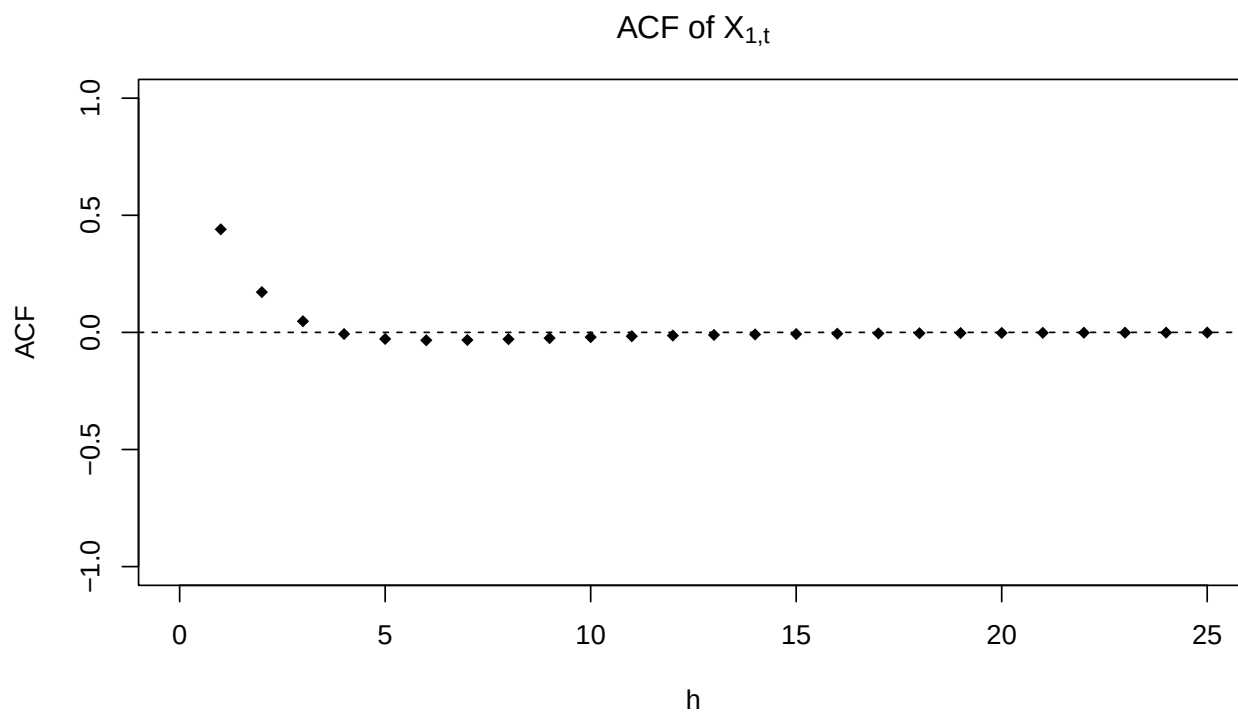
```
Rhoh[3,,] #h=3
```

```
##           [,1]      [,2]
## [1,]  0.0476000  0.3238172
## [2,] -0.0433232  0.5120000
```

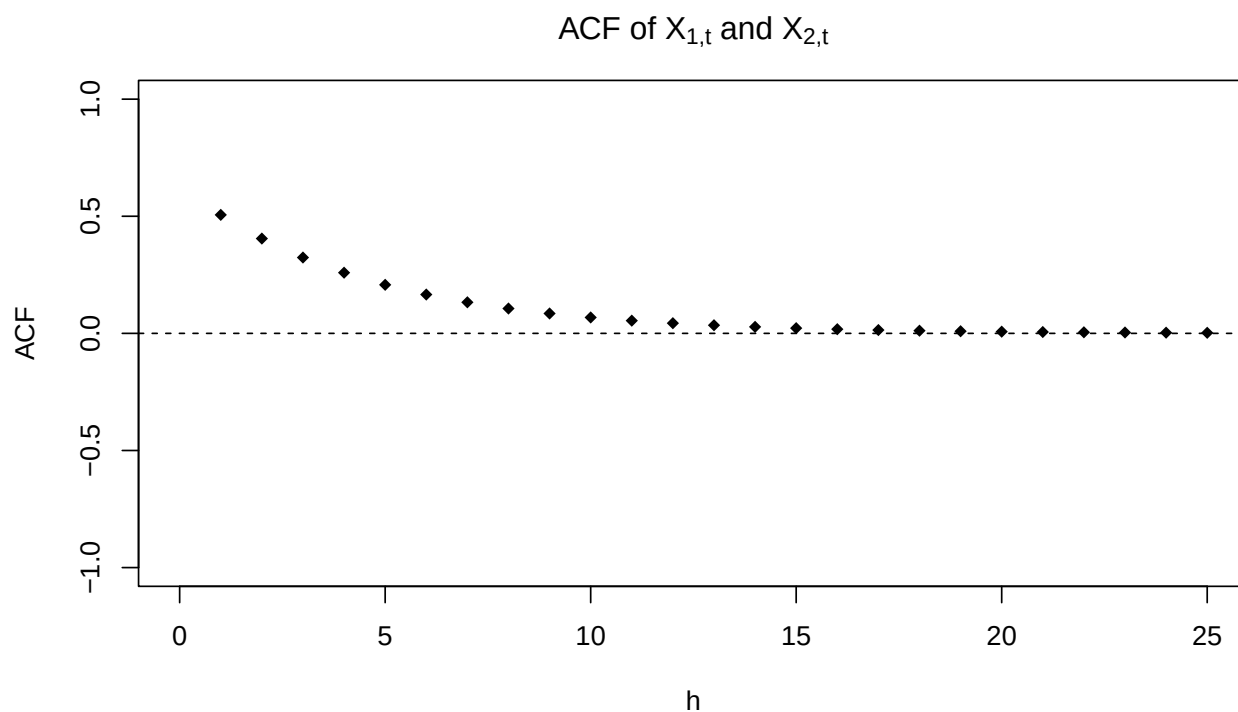
```
Rhoh[4,,] #h=4
```

```
##           [,1]      [,2]
## [1,] -0.00692000  0.2590538
## [2,] -0.07023419  0.4096000
```

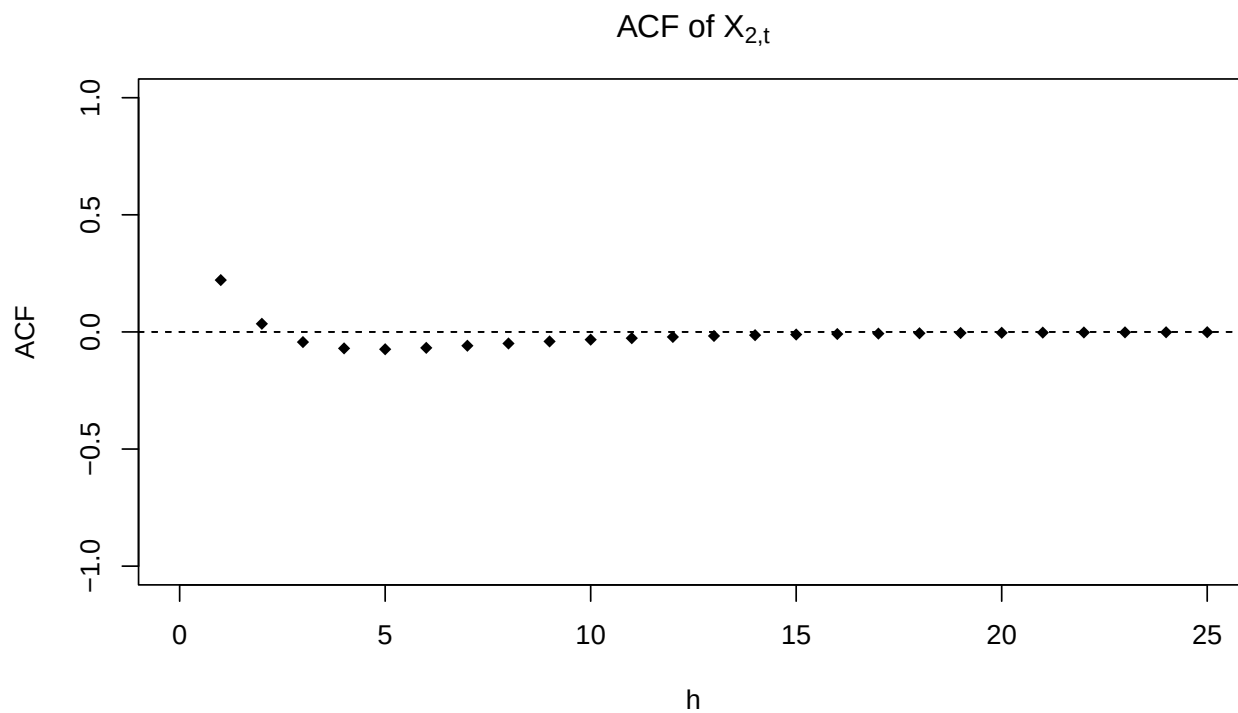
```
par(mar=c(4,4,3,2))
plot(Rhoh[1:25,1,1],pch=18,xlim=range(0,25),ylim=range(-1,1),xlab='h',ylab='ACF')
title(expression(paste('ACF of ',X[paste(1,"",t)]))); abline(h=0,lty=2)
```



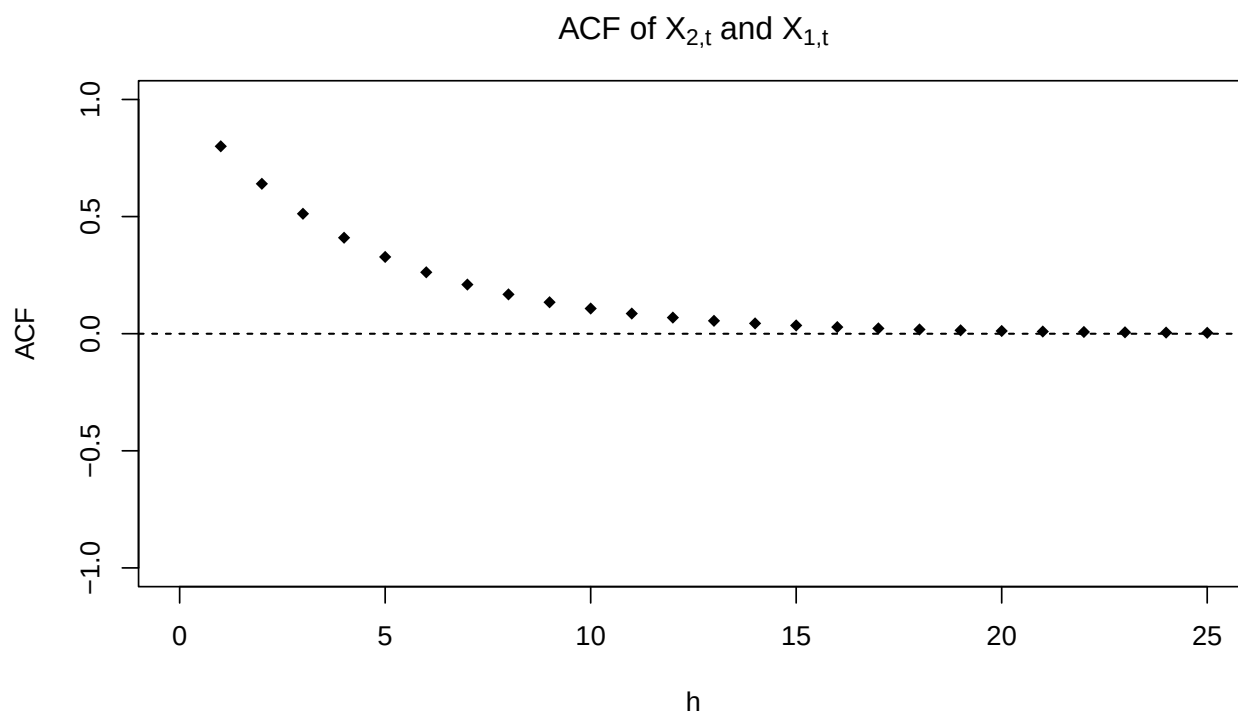
```
plot(Rhoh[1:25,1,2],pch=18,xlim=range(0,25),ylim=range(-1,1),xlab='h',ylab='ACF')
title(expression(paste('ACF of ',X[paste(1,"",t)],' and ',X[paste(2,"",t)])))
abline(h=0,lty=2)
```



```
plot(Rhoh[1:25,2,1],pch=18,xlim=range(0,25),ylim=range(-1,1),xlab='h',ylab='ACF')
abline(h=0,lty=2)
title(expression(paste('ACF of ',X[paste(2,"",t)]))); abline(h=0,lty=2)
```



```
plot(Rhoh[1:25,2,2],pch=18,xlim=range(0,25),ylim=range(-1,1),xlab='h',ylab='ACF')
abline(h=0,lty=2)
title(expression(paste('ACF of ',X[paste(2,"",t)], ' and ',X[paste(1,"",t)])))
abline(h=0,lty=2)
```



Recall that

$$\rho_{ij}(h) = \text{Corr}[X_{i,t}, X_{j,t-h}]$$

and

$$\rho_{ji}(h) = \text{Corr}[X_{j,t}, X_{i,t-h}] = \text{Corr}[X_{i,t-h}, X_{j,t}] = \text{Corr}[X_{i,t}, X_{j,t+h}] = \rho_{ij}(-h).$$

For this process, we observe that

$$\rho_{11}(1) = 0.44$$

$$\rho_{12}(1) = 0.5059644$$

$$\rho_{21}(1) = 0.2213594$$

$$\rho_{22}(1) = 0.8$$