

MATH 545: INTRODUCTION TO TIME SERIES ANALYSIS

WINTER 2025

VECTOR AUTOREGRESSION MODELS

Consider the model

$$\begin{aligned} X_{1,t} &= \phi_{10} + \phi_{11}X_{1,t-1} + \phi_{12}X_{2,t-1} + Z_{1,t} \\ X_{2,t} &= \phi_{20} + \phi_{21}X_{1,t-1} + \phi_{22}X_{2,t-1} + Z_{2,t} \end{aligned}$$

The elements of

$$\Phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

define the dependence of current elements of $\mathbf{X}_t$ on past elements. Let

$$\Sigma = \mathbf{L}\mathbf{G}\mathbf{L}^\top$$

be a Cholesky decomposition where $\mathbf{G}$ is a diagonal matrix, and $\mathbf{L}$ is a lower-triangular matrix, where the diagonal elements of $\mathbf{L}$ are all equal to 1. Define $\mathbf{U}_t = \mathbf{L}^{-1}\mathbf{Z}_t$. Then

$$\mathbb{E}[\mathbf{U}_t] = \mathbf{0} \quad \text{Var}[\mathbf{U}_t] = \mathbf{G}$$

so that the elements of $\mathbf{U}_t$ are uncorrelated. Then

$$\mathbf{L}^{-1}\mathbf{X}_t = \mathbf{L}^{-1}\phi_0 + \mathbf{L}^{-1}\Phi\mathbf{X}_{t-1} + \mathbf{L}^{-1}\mathbf{Z}_t$$

so that

$$\mathbf{L}^{-1}\mathbf{X}_t = \phi^* + \Phi^*\mathbf{X}_{t-1} + \mathbf{U}_t$$

say. Thus, as $\mathbf{L}$ is lower-triangular, the K th row of this vector equation is

$$X_{K,t} + \sum_{k=1}^{K-1} L_{Kk}X_{k,t} = \phi_{K0}^* + \sum_{k=1}^K \phi_{Kk}^*X_{k,t-1} + U_{K,t}$$

As the elements of $\mathbf{U}_t$ are uncorrelated, this equation shows explicitly the concurrent dependence of $X_{K,t}$ on $X_{k,t}$, for $k = 1, \dots, K-1$. A similar expression can be found for each of the elements of $\mathbf{X}_t$.

Example: Let

$$\begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix} + \begin{bmatrix} 0.2 & 0.3 \\ -0.6 & 1.1 \end{bmatrix} \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \end{bmatrix} + \begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix}$$

with

$$\Sigma = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Then

$$\mathbf{L}^{-1} = \begin{bmatrix} 1.0 & 0.0 \\ -0.5 & 1.0 \end{bmatrix} \quad \mathbf{G} = \mathbf{L}^{-1}\Sigma(\mathbf{L}^\top)^{-1} = \begin{bmatrix} 2.0 & 0.0 \\ 0.0 & 0.5 \end{bmatrix}$$

Using the above structural equation formulation:

$$X_{2,t} = 0.3 + 0.5X_{1,t} - 0.7X_{1,t-1} + 0.95X_{2,t-1} + U_{2,t}$$

and

$$X_{1,t} = -0.2 + 1.0X_{2,t} - 0.8X_{2,t-1} + 0.8X_{1,t-1} + U_{1,t}$$

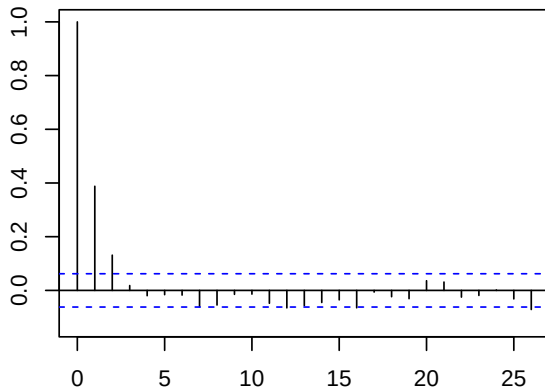
where $U_{2,t} \sim \mathcal{N}(0, 0.5)$ and $U_{1,t} \sim \mathcal{N}(0, 1)$ are independent random variables.

```

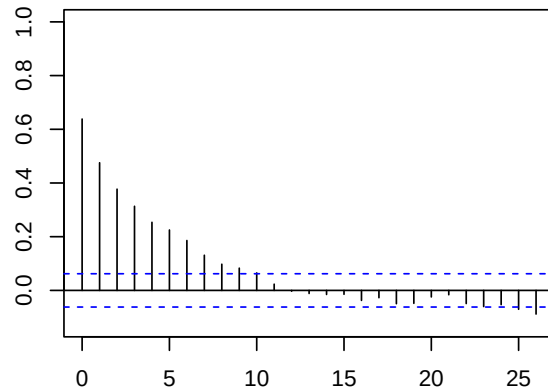
library(mvtnfast)
n<-1000
Sigma<-matrix(c(2,1,1,1),nrow=2,byrow=T)
Phi<-matrix(c(0.2,0.3,-0.6,1.1),nrow=2,byrow=T)
Phi0<-c(0.2,0.4)
Zt<-rmvn(n,c(0,0),Sigma)
Xt<-matrix(0,nrow=n,ncol=2)
for(i in 2:n){
  Xt[i,]<-Phi0+Phi %*% Xt[i-1,] + Zt[i,]
}
par(mar=c(2,2,4,4))
acf(Xt)

```

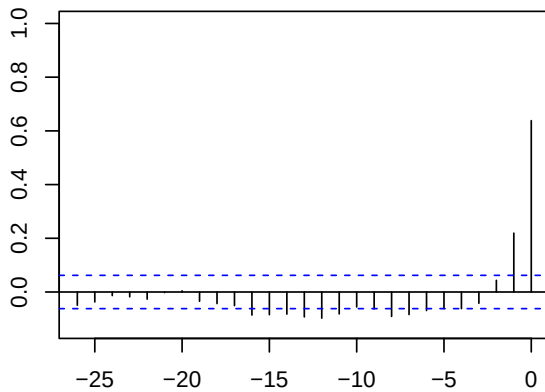
Series 1



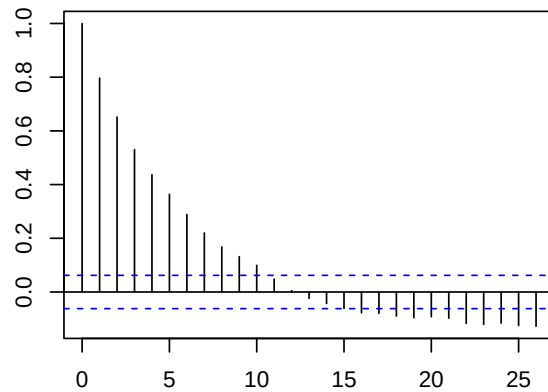
Series 1 & Series 2



Series 2 & Series 1



Series 2



Now for the usual Cholesky decomposition, we have

$$\mathbf{C}\mathbf{C}^\top = \Sigma$$

where $\mathbf{C}$ is lower triangular, but here we want

$$\mathbf{L}\mathbf{G}\mathbf{L}^\top = \Sigma.$$

The diagonal elements of $\mathbf{C}$ are square roots of the diagonal elements of Σ . Hence $\mathbf{C} = \mathbf{L}\mathbf{G}^{1/2}$ so that $\mathbf{L} = \mathbf{C}\mathbf{G}^{-1/2}$. We then have

$$\text{Var}[\mathbf{U}_t] = \text{Var}[\mathbf{L}^{-1}\mathbf{Z}_t] = \mathbf{L}^{-1}\text{Var}[\mathbf{Z}_t]\mathbf{L}^{-\top} = \mathbf{L}^{-1}\Sigma\mathbf{L}^{-\top} = \mathbf{G}.$$

where $\mathbf{L}^{-\top}$ is the transpose of $\mathbf{L}^{-1}$.

```

Sigma

##          [,1] [,2]
## [1,]      2    1
## [2,]      1    1

Cmat<-t(chol(Sigma))
Cmat

##          [,1] [,2]
## [1,] 1.4142136 0.0000000
## [2,] 0.7071068 0.7071068

Gmat.root<-diag(diag(Cmat))
Gmat.root

##          [,1] [,2]
## [1,] 1.414214 0.0000000
## [2,] 0.000000 0.7071068

Lmat<-Cmat %*% solve(Gmat.root)
Gmat<-Gmat.root^2
Lmat

##          [,1] [,2]
## [1,]    1.0    0
## [2,]    0.5    1

Gmat

##          [,1] [,2]
## [1,]      2  0.0
## [2,]      0  0.5

Linv<-solve(Lmat)
Linv

##          [,1] [,2]
## [1,]    1.0    0
## [2,]   -0.5    1

```

We can then compute

$$\phi^* = \mathbf{L}^{-1} \phi_0$$

```

PhiOstar<-Linv %*% Phi0
PhiOstar

##          [,1]
## [1,]    0.2
## [2,]    0.3

```

and

$$\Phi^* = \mathbf{L}^{-1} \Phi$$

```

Phistar<-Linv %*% Phi
Phistar

##          [,1] [,2]
## [1,]    0.2 0.30
## [2,]   -0.7 0.95

```

To obtain the structural equation for $X_{2,t}$, we note that

$$\mathbf{L}^{-1}\mathbf{X}_t = \boldsymbol{\phi}^* + \boldsymbol{\Phi}^*\mathbf{X}_{t-1} + \mathbf{U}_t$$

becomes

$$\begin{bmatrix} 1 & 0 \\ -0.5 & 1 \end{bmatrix} \begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix} + \begin{bmatrix} 0.2 & 0.3 \\ -0.7 & 0.95 \end{bmatrix} \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \end{bmatrix} + \begin{bmatrix} U_{1,t} \\ U_{2,t} \end{bmatrix}$$

from which we can deduce via the second equation that

$$-0.5X_{1,t} + X_{2,t} = 0.3 - 0.7X_{1,t-1} + 0.95X_{2,t-1} + U_{2,t}$$

that is

$$X_{2,t} = 0.5X_{1,t} + 0.3 - 0.7X_{1,t-1} + 0.95X_{2,t-1} + U_{2,t}.$$

Note that from the first equation, we have

$$X_{1,t} = 0.2 + 0.2X_{1,t-1} + 0.3X_{2,t-1} + U_{1,t}$$

which shows the dependence of $\{X_{1,t}\}$ on lagged values only.

To obtain the structural equation for $\{X_{1,t}\}$, we swap the positions of the two variables: we have

$$\begin{bmatrix} X_{2,t} \\ X_{1,t} \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix} + \begin{bmatrix} 1.1 & -0.6 \\ 0.3 & 0.2 \end{bmatrix} \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \end{bmatrix} + \begin{bmatrix} Z_{2,t} \\ Z_{1,t} \end{bmatrix}$$

with now

$$\Sigma = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

```
Sigma<-matrix(c(1,1,1,2),nrow=2,byrow=T)
Phi<-matrix(c(1.1,-0.6,0.3,0.2),nrow=2,byrow=T)
Phi0<-c(0.4,0.2)
```

```
Sigma
```

```
##      [,1] [,2]
## [1,]    1    1
## [2,]    1    2
```

```
Cmat<-t(chol(Sigma))
Cmat
```

```
##      [,1] [,2]
## [1,]    1    0
## [2,]    1    1
```

```
Gmat.root<-diag(diag(Cmat))
Gmat.root
```

```
##      [,1] [,2]
## [1,]    1    0
## [2,]    0    1
```

```
Lmat<-Cmat %*% solve(Gmat.root)
Gmat<-Gmat.root^2
Lmat
```

```
##      [,1] [,2]
## [1,]    1    0
## [2,]    1    1
```

```
Gmat
```

```
##      [,1] [,2]
## [1,]    1    0
## [2,]    0    1

Linv<-solve(Lmat)
Linv

##      [,1] [,2]
## [1,]    1    0
## [2,]   -1    1

PhiOstar<-Linv %*% Phi0
PhiOstar

##      [,1]
## [1,]  0.4
## [2,] -0.2

Phistar<-Linv %*% Phi
Phistar

##      [,1] [,2]
## [1,]  1.1 -0.6
## [2,] -0.8  0.8
```

so that

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} X_{2,t} \\ X_{1,t} \end{bmatrix} = \begin{bmatrix} 0.4 \\ -0.2 \end{bmatrix} + \begin{bmatrix} 1.1 & -0.6 \\ -0.8 & 0.8 \end{bmatrix} \begin{bmatrix} X_{2,t-1} \\ X_{1,t-1} \end{bmatrix} + \begin{bmatrix} U_{2,t} \\ U_{1,t} \end{bmatrix}$$

yielding the structural equation

$$X_{1,t} = -0.2 + X_{2,t} + 0.8X_{1,t-1} - 0.8X_{2,t-1} + U_{1,t}$$

where $\text{Var}[U_{1,t}] = 1$, as note that here

$$\mathbf{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$