

MATH 545: INTRODUCTION TO TIME SERIES ANALYSIS

WINTER 2025

ARMA PROCESSES

An ARMA(p, q) process $\{X_t\}$ is defined for $\{Z_t\} \sim \text{WN}(0, \sigma_Z^2)$ and integers $p, q \geq 0$ as the process defined implicitly by

$$X_t = \sum_{j=1}^p \phi_j X_{t-j} + \sum_{j=0}^q \theta_j Z_{t-j} \quad \forall t \quad (1)$$

with $\theta_0 \equiv 1$, or equivalently

$$\Phi(B)X_t = \Theta(B)Z_t \quad (2)$$

where

$$\Phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p \quad \Theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q.$$

Using the factorization method, we write

$$\Phi(B) = \prod_{j=1}^p (1 - \xi_j B) \quad \Theta(B) = \prod_{j=1}^q (1 - \omega_j B)$$

and assume that there are no common factors in the two products (otherwise we could cancel terms on both sides, and reduce the orders of the two polynomials). We have shown that the process is stationary and causal if and only if $|\xi_j| < 1, j = 1, \dots, p$, in which case we know that the linear process representation

$$X_t = \Psi(B)Z_t = \left(\sum_{j=0}^{\infty} \psi_j B^j \right) Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

exists with $\sum_{j=0}^{\infty} |\psi_j| < \infty$.

We have also seen some general methods of computation. To identify the ψ_j , we can plug the proposed solution into (2) and equate coefficients:

$$\Phi(B)\Psi(B) = \Theta(B)$$

or

$$(1 + \phi_1 B + \cdots + \phi_p B^p)(\psi_0 + \psi_1 B + \psi_2 B^2 + \cdots) = 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q.$$

Equating the coefficient of B^j , we have

$$\psi_j - \phi_1 \psi_{j-1} - \cdots - \phi_p \psi_{j-p} = \theta_j \quad j = 0, 1, 2, \dots$$

where $\theta_j = 0, j > q$, and $\psi_j = 0, j < 0$. Rearranging, we get the recursive formula

$$\psi_j = \theta_j + \sum_{k=1}^p \phi_k \psi_{j-k} \quad j = 1, 2, \dots$$

with $\psi_0 = 1$, or more precisely

$$\psi_j = \begin{cases} \theta_j + \sum_{k=1}^j \phi_k \psi_{j-k} & 0 \leq j < \max\{p, q+1\} \\ \sum_{k=1}^p \phi_k \psi_{j-k} & j \geq \max\{p, q+1\} \end{cases}$$

Note that this is merely a system of linear recurrence relations for which we have seen methods for constructing the exact solution using eigen decompositions.

To illustrate this recursion, consider the ARMA(3, 1) model

$$X_t = 0.5X_{t-1} + 0.2X_{t-2} - 0.2X_{t-3} + Z_t + 2Z_{t-1}$$

```
p<-3;q<-1
lmax<-200
phi.vec<-theta.vec<-rep(0,lmax+1)
phi.vec[1:p]<-c(0.5,0.2,-0.2)
theta.vec[1:q]<-c(2)
true.psi.vec<-ARMAtoMA(ar=phi.vec[1:p],ma=theta.vec[1:q],lag.max=lmax) #True values
psi.vec<-rep(0,lmax+1)
psi.vec[0+1]<-1
for(j in 1:lmax){
  kvec<-1:min(j,p)
  psi.vec[j+1]<-theta.vec[j]+sum(phi.vec[kvec]*psi.vec[j-kvec+1])
}
cbind(1:10,psi.vec[1:10+1],true.psi.vec[1:10])
```

	[,1]	[,2]	[,3]
+ [1,]	1	2.50000000	2.50000000
+ [2,]	2	1.45000000	1.45000000
+ [3,]	3	1.02500000	1.02500000
+ [4,]	4	0.30250000	0.30250000
+ [5,]	5	0.06625000	0.06625000
+ [6,]	6	-0.11137500	-0.11137500
+ [7,]	7	-0.10293750	-0.10293750
+ [8,]	8	-0.08699375	-0.08699375
+ [9,]	9	-0.04180937	-0.04180937
+ [10,]	10	-0.01771594	-0.01771594

The other key computation is for the autocovariance sequence. The technique here is to multiply both sides of (1) by X_{t-h} , and then take expectations:

$$\mathbb{E}[X_t X_{t-h}] = \sum_{j=1}^p \phi_j \mathbb{E}[X_{t-j} X_{t-h}] + \sum_{j=0}^q \theta_j \mathbb{E}[Z_{t-j} X_{t-h}]. \quad (3)$$

Now

$$\mathbb{E}[Z_{t-j} X_{t-h}] = \sum_{k=0}^{\infty} \psi_k \mathbb{E}[Z_{t-j} Z_{t-h-k}] = \sigma_Z^2 \psi_{j-h}$$

Thus we have for $h \geq 0$

$$\gamma_X(h) - \sum_{j=1}^p \phi_j \gamma_X(h-j) = \sigma_Z^2 \sum_{j=h}^q \theta_j \psi_{j-h} \equiv \sigma_Z^2 \sum_{j=0}^{q-h} \theta_{j+h} \psi_j \quad (4)$$

as, recall, $\psi_{j-h} = 0$ if $j < h$. From (4), we can obtain a linear system of equations, and then deduce the remaining values in the sequence from the fact that

$$\gamma_X(h) = \sum_{j=1}^p \phi_j \gamma_X(h-j). \quad (5)$$

whenever $h \geq m = \max\{p, q+1\}$. A rule of thumb is

- if $p \geq q+1$, inspect equations for lags $h = 0, 1, \dots, p$;
- if $p < q+1$, inspect equations for lags $h = 0, 1, \dots, q$;

EXAMPLE: For the ARMA(3, 1) model

$$X_t = 0.5X_{t-1} + 0.2X_{t-2} - 0.2X_{t-3} + Z_t + 2Z_{t-1}$$

we have $m = \max\{p, q + 1\} = 3$ with $p > q + 1$, and therefore inspect $h = 0, 1, 2, 3$. We have the system

$$\begin{aligned} h = 0 : \quad \gamma_X(0) &= \phi_1\gamma_X(1) + \phi_2\gamma_X(2) + \phi_3\gamma_X(3) + \sigma_Z^2(1 + \theta_1\psi_1) \\ h = 1 : \quad \gamma_X(1) &= \phi_1\gamma_X(0) + \phi_2\gamma_X(1) + \phi_3\gamma_X(2) + \sigma_Z^2\theta_1 \\ h = 2 : \quad \gamma_X(2) &= \phi_1\gamma_X(1) + \phi_2\gamma_X(0) + \phi_3\gamma_X(1) \\ h = 3 : \quad \gamma_X(3) &= \phi_1\gamma_X(2) + \phi_2\gamma_X(1) + \phi_3\gamma_X(0) \end{aligned}$$

with $\phi_1 = 0.5, \phi_2 = 0.2, \phi_3 = -0.2$ and $\theta_1 = 2$. We re-write this as

$$\begin{bmatrix} 1 & -\phi_1 & -\phi_2 & -\phi_3 \\ -\phi_1 & 1 - \phi_2 & -\phi_3 & 0 \\ -\phi_2 & -\phi_1 - \phi_3 & 1 & 0 \\ -\phi_3 & -\phi_2 & -\phi_1 & 1 \end{bmatrix} \begin{bmatrix} \gamma_X(0) \\ \gamma_X(1) \\ \gamma_X(2) \\ \gamma_X(3) \end{bmatrix} = \begin{bmatrix} \sigma_Z^2(1 + \theta_1\psi_1) \\ \sigma_Z^2\theta_1 \\ 0 \\ 0 \end{bmatrix}$$

or $\mathbf{Ax} = \mathbf{b}$, which we want to solve for $\mathbf{x}$.

```
p<-3;q<-1;m<-max(p,q+1)
phi.vec<-theta.vec<-rep(0,lmax+1)
phi.vec[1:p]<-c(0.5,0.2,-0.2)
theta.vec[1:q]<-c(2)
true.gammaX<-ARMAacf(ar=phi.vec[1:p],ma=theta.vec[1:q],lag.max=lmax) #True values
Amat<-matrix(0,4,4)
Amat[1,]<-c(1,-phi.vec[1:p])
Amat[2,]<-c(-phi.vec[1],1-phi.vec[2],-phi.vec[3],0)
Amat[3,]<-c(-phi.vec[2],-phi.vec[1]-phi.vec[3],1,0)
Amat[4,]<-c(-phi.vec[3],-phi.vec[2],-phi.vec[1],1)
bvec<-c(1+theta.vec[1]*psi.vec[1+1], theta.vec[1],0,0)
gammaX<-solve(Amat,bvec)
gammaX/gammaX[1]

+ [1] 1.0000000 0.7557003 0.4267101 0.1644951

true.gammaX[1:4]

+           0           1           2           3
+ 1.0000000 0.7557003 0.4267101 0.1644951
```

Then using (5), we can compute the remaining values (out to $h = 9$ say):

```
for(h in (m+1):lmax){
  gammaX<-c(gammaX,sum(phi.vec*gammaX[h-(1:p)+1]))
}
cbind(gammaX[1:10]/gammaX[1],true.gammaX[1:10]) #Comparison of equivalent computations

+           [,1]           [,2]
+ 0 1.000000000 1.000000000
+ 1 0.755700326 0.755700326
+ 2 0.426710098 0.426710098
+ 3 0.164495114 0.164495114
+ 4 0.016449511 0.016449511
+ 5 -0.044218241 -0.044218241
+ 6 -0.051718241 -0.051718241
+ 7 -0.037992671 -0.037992671
+ 8 -0.020496336 -0.020496336
+ 9 -0.007503054 -0.007503054
```

EXAMPLE: For the ARMA(2, 4) model

$$X_t = X_{t-1} - 0.25X_{t-2} + Z_t + Z_{t-1} + 3Z_{t-2} - Z_{t-3} + 2Z_{t-4}$$

we have $m = \max\{p, q + 1\} = 5$ with $p < q + 1$, and therefore inspect $h = 0, 1, 2, 3, 4$. We have the system

$$\begin{aligned} h = 0 : \quad \gamma_X(0) &= \phi_1\gamma_X(1) + \phi_2\gamma_X(2) + \sigma_Z^2(1 + \theta_1\psi_1 + \theta_2\psi_2 + \theta_3\psi_3 + \theta_4\psi_4) \\ h = 1 : \quad \gamma_X(1) &= \phi_1\gamma_X(0) + \phi_2\gamma_X(1) + \sigma_Z^2(\theta_1 + \theta_2\psi_1 + \theta_3\psi_2 + \theta_4\psi_3) \\ h = 2 : \quad \gamma_X(2) &= \phi_1\gamma_X(1) + \phi_2\gamma_X(0) + \sigma_Z^2(\theta_2 + \theta_3\psi_1 + \theta_4\psi_2) \\ h = 3 : \quad \gamma_X(3) &= \phi_1\gamma_X(2) + \phi_2\gamma_X(1) + \sigma_Z^2(\theta_3 + \theta_4\psi_1) \\ h = 4 : \quad \gamma_X(4) &= \phi_1\gamma_X(3) + \phi_2\gamma_X(2) + \sigma_Z^2\theta_4 \end{aligned}$$

with $\phi_1 = 1, \phi_2 = -0.25$, and $\theta_1 = 1, \theta_2 = 3, \theta_3 = -1, \theta_4 = 2$. We re-write this as

$$\begin{bmatrix} 1 & -\phi_1 & -\phi_2 & 0 & 0 \\ -\phi_1 & 1 - \phi_2 & 0 & 0 & 0 \\ -\phi_2 & -\phi_1 & 1 & 0 & 0 \\ 0 & -\phi_2 & -\phi_1 & 1 & 0 \\ 0 & 0 & -\phi_2 & -\phi_1 & 1 \end{bmatrix} \begin{bmatrix} \gamma_X(0) \\ \gamma_X(1) \\ \gamma_X(2) \\ \gamma_X(3) \\ \gamma_X(4) \end{bmatrix} = \begin{bmatrix} \sigma_Z^2(1 + \theta_1\psi_1 + \theta_2\psi_2 + \theta_3\psi_3 + \theta_4\psi_4) \\ \sigma_Z^2(\theta_1 + \theta_2\psi_1 + \theta_3\psi_2 + \theta_4\psi_3) \\ \sigma_Z^2(\theta_2 + \theta_3\psi_1 + \theta_4\psi_2) \\ \sigma_Z^2(\theta_3 + \theta_4\psi_1) \\ \sigma_Z^2\theta_4 \end{bmatrix}$$

or $\mathbf{Ax} = \mathbf{b}$, which we want to solve for $\mathbf{x}$.

```
p<-2;q<-4;m<-max(p,q+1)
phi.vec<-theta.vec<-rep(0,lmax+1)
phi.vec[1:p]<-c(1,-0.25)
theta.vec[1:q]<-c(1,3,-1,2)
true.psi.vec<-ARMAtoMA(ar=phi.vec[1:p],ma=theta.vec[1:q],lag.max=lmax) #True values
psi.vec<-rep(0,lmax+1)
psi.vec[0+1]<-1
for(j in 1:lmax){
  kvec<-1:min(j,p)
  psi.vec[j+1]<-theta.vec[j]+sum(phi.vec[kvec]*psi.vec[j-kvec+1])
}
cbind(1:10,psi.vec[1:10+1],true.psi.vec[1:10]) #Comparison of equivalent computations

+      [,1]      [,2]      [,3]
+ [1,]  1 2.0000000 2.0000000
+ [2,]  2 4.7500000 4.7500000
+ [3,]  3 3.2500000 3.2500000
+ [4,]  4 4.0625000 4.0625000
+ [5,]  5 3.2500000 3.2500000
+ [6,]  6 2.2343750 2.2343750
+ [7,]  7 1.4218750 1.4218750
+ [8,]  8 0.8632812 0.8632812
+ [9,]  9 0.5078125 0.5078125
+ [10,] 10 0.2919922 0.2919922

true.gammaX<-ARMAacf(ar=phi.vec[1:p],ma=theta.vec[1:q],lag.max=lmax) #True values
Amat<-matrix(0,5,5)
Amat[1,]<-c(1,-phi.vec[1:p],0,0)
Amat[2,]<-c(-phi.vec[1],1-phi.vec[2],0,0,0)
Amat[3,]<-c(-phi.vec[2],-phi.vec[1],1,0,0)
Amat[4,]<-c(0,-phi.vec[2],-phi.vec[1],1,0)
Amat[5,]<-c(0,0,-phi.vec[2],-phi.vec[1],1)
bvec<-c(sum(c(1,theta.vec[1:4])*psi.vec[1:5]),sum(theta.vec[1:4]*psi.vec[1:4]),
        sum(theta.vec[2:4]*psi.vec[1:3]),sum(theta.vec[3:4]*psi.vec[1:2]),
        sum(theta.vec[4]*psi.vec[1]))
gammaX<-solve(Amat,bvec)
gammaX/gammaX[1]
```

```
+ [1] 1.0000000 0.8954545 0.7886364 0.6056818 0.4357955

true.gammaX[1:5]

+           0           1           2           3           4
+ 1.0000000 0.8954545 0.7886364 0.6056818 0.4357955
```

Then using (5), we can compute the remaining values (out to $h = 9$ say):

```
for(h in m:lmax){
  gammaX<-c(gammaX,sum(phi.vec[1:p]*gammaX[h-(1:p)+1]))
}
cbind(gammaX[1:10]/gammaX[1],true.gammaX[1:10]) #Comparison of equivalent computations

+           [,1]           [,2]
+ 0 1.00000000 1.00000000
+ 1 0.89545455 0.89545455
+ 2 0.78863636 0.78863636
+ 3 0.60568182 0.60568182
+ 4 0.43579545 0.43579545
+ 5 0.28437500 0.28437500
+ 6 0.17542614 0.17542614
+ 7 0.10433239 0.10433239
+ 8 0.06047585 0.06047585
+ 9 0.03439276 0.03439276
```

For a simulation from this process:

```
set.seed(234)
X<-arima.sim(n=200,model=list(ar=phi.vec[1:p],ma=theta.vec[1:q]))
par(mar=c(4,4,0,1))
plot(X,type='l',ylim=range(-30,30));points(X,pch=19,cex=0.6)
```

