

MATH 545: INTRODUCTION TO TIME SERIES ANALYSIS

WINTER 2025

STATIONARY, CAUSAL AND INVERTIBLE ARMA SOLUTIONS

$\{X_t\}$ is an ARMA(p, q) process if it is stationary and satisfies

$$\Phi(B)X_t = \Theta(B)Z_t \quad (1)$$

where $\{Z_t\} \sim \text{WN}(0, \sigma_Z^2)$, and

$$\begin{aligned} \Phi(z) &= 1 - \phi_1 z - \cdots - \phi_p z^p = \prod_{j=1}^p (1 - \xi_j z) \\ \Theta(z) &= 1 + \theta_1 z + \cdots + \theta_q z^q = \prod_{j=1}^q (1 + \omega_j z) \end{aligned}$$

That is

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q}$$

or equivalently

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q}$$

Reciprocal roots: For the AR component, the parameters $\xi_1, \dots, \xi_p$ are the reciprocal roots of the equation

$$\Phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p = 0$$

that is, the reciprocals of the roots, η_j , of the equation

$$\prod_{j=1}^p (\eta_j - z) = 0$$

To see this, note that $(1 - \xi z) = \xi(\xi^{-1} - z) = \xi(\eta - z)$ so therefore

$$\prod_{j=1}^p (1 - \xi_j z) = \left(\prod_{j=1}^p \xi_j \right) \prod_{j=1}^p (\xi_j^{-1} - z).$$

The function `polyroot` in R can be used to find the reciprocal roots from an arbitrary polynomial.

Example: Consider the polynomial

$$\Phi(z) = 1 - 2z + \frac{1}{2}z^2 + \frac{2}{3}z^3$$

to define the AR(3) model

$$X_t - 2X_{t-1} + \frac{1}{2}X_{t-2} + \frac{2}{3}X_{t-3} = Z_t$$

or equivalently

$$X_t = 2X_{t-1} - \frac{1}{2}X_{t-2} - \frac{2}{3}X_{t-3} + Z_t.$$

```
covec<-c(1,-2,1/2,2/3) #Enter the coefficient of 1, z, z^2, z^3
polyroot(covec)

## [1] 0.7854979+0.1710774i 0.7854979-0.1710774i -2.3209957+0.0000000i

1/polyroot(covec)

## [1] 1.2154248-0.2647132i 1.2154248+0.2647132i -0.4308496+0.0000000i
```

The three reciprocal roots are

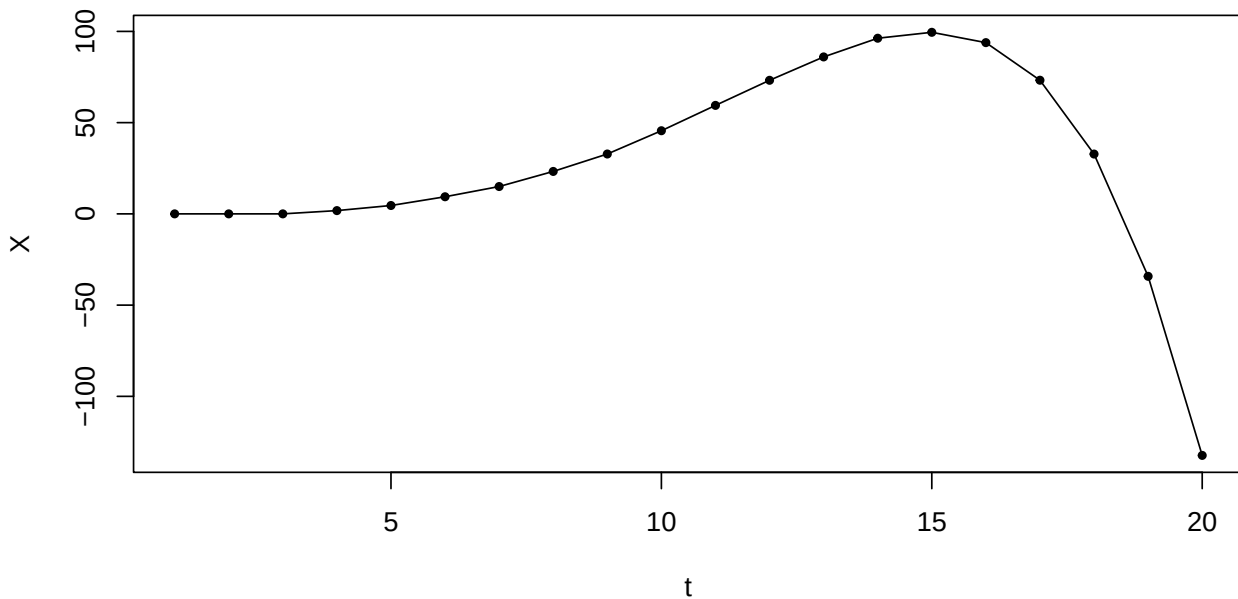
$$1.2154 - 0.2647i, 1.2154 + 0.2647i, -0.4308 + 0i$$

that is, there is one real reciprocal root, -0.4308 , and two complex reciprocal roots, that appear in a complex conjugate pair. We can check the modulus of the three roots using the function `Mod`:

```
Mod(1/polyroot(covec))
## [1] 1.2439174 1.2439174 0.4308496
```

This tells us that the complex-valued reciprocal roots have modulus 1.2439174 , and so no stationary AR(3) component can be defined with these coefficients. We can example the AR(3) process defined in this way.

```
set.seed(23)
n<-20
Z<-rnorm(n)
X<-rep(0,n)
for(i in 4:n){
  X[i]<-2*X[i-1]-(1/2)*X[i-2]-(2/3)*X[i-3]+Z[i]
}
par(mar=c(4,4,2,2))
plot(X,type='l',xlab='t')
points(X,pch=19,cex=0.6)
```



The system exhibits explosive behaviour.

We can construct a stationary process by finding the coefficients corresponding to a particular set of reciprocal roots. For example, if we consider entirely real reciprocal roots

$$\xi_1 = -0.1, \quad \xi_2 = 0.25, \quad \xi_3 = 0.75 \quad \implies \quad \eta_1 = -10, \quad \eta_2 = 4, \quad \eta_3 = \frac{4}{3}$$

so that the AR(3) polynomial

$$(z - \eta_1)(z - \eta_2)(z - \eta_3)$$

can be computed directly using the `polynom` package. The `polynom` package allows multiplication of polynomials, in this case, to perform the calculation

$$(10 + z)(-4 + z)(-4/3 + z)$$

```

library(polynom)
t1 <- polynomial(c(10,1))      #First term: 10+z
t2 <- polynomial(c(-4,1))      #Second term: -4+z
t3 <- polynomial(c(-4/3,1))    #Third term: -4/3+z
(t1*t2)*t3

## 53.33333 - 48*x + 4.666667*x^2 + x^3

avec<-coefficients((t1*t2)*t3)
polyroot(avec)                 #Roots

## [1] 1.333333+0.000000e+00i 4.000000+4.694904e-15i -10.000000-4.694904e-15i

polyroot(rev(avec))            #Reciprocal roots

## [1] -0.10-6.262029e-18i 0.25+1.064545e-17i 0.75-4.383420e-18i

covec<-avec/avec[1]
covec

## [1] 1.00000 -0.90000 0.08750 0.01875

```

Thus the AR(3) model

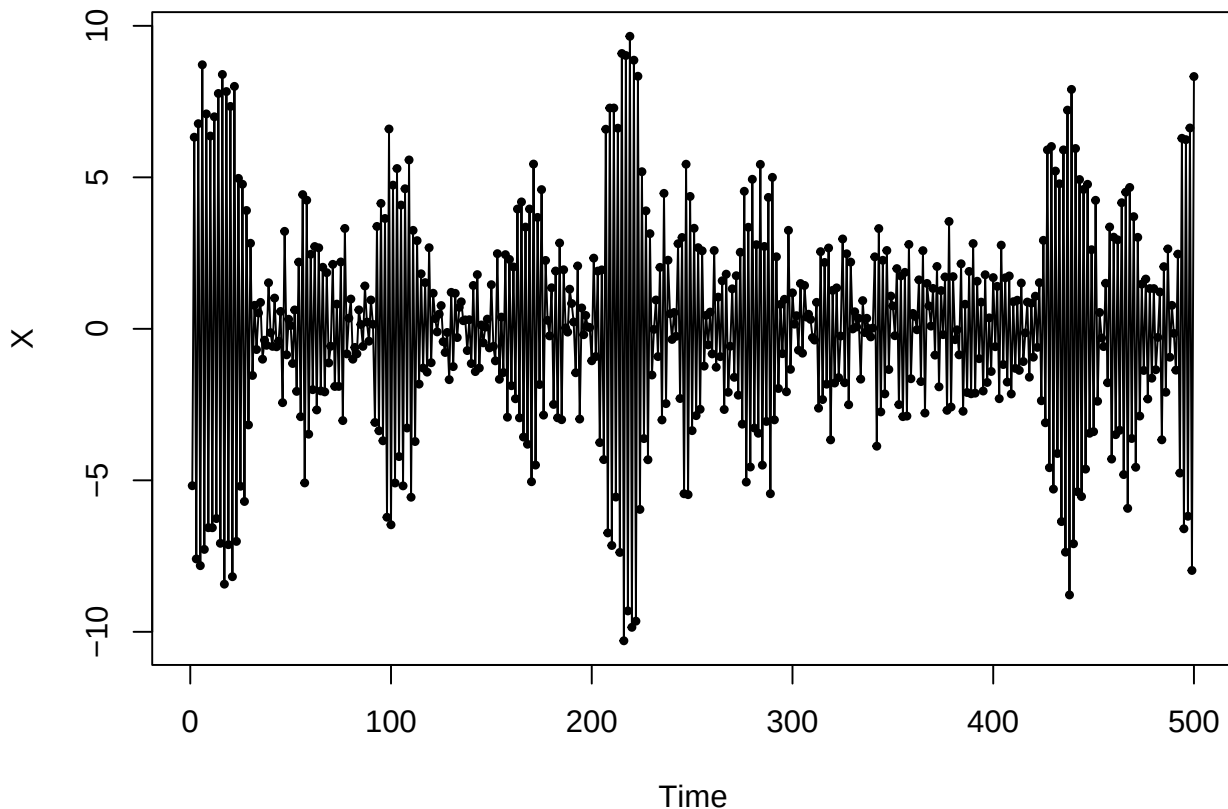
$$X_t = -0.9X_{t-1} + 0.0875X_{t-2} + 0.01875X_{t-3} + Z_t$$

should be stationary. We use the `arima.sim` to check this:

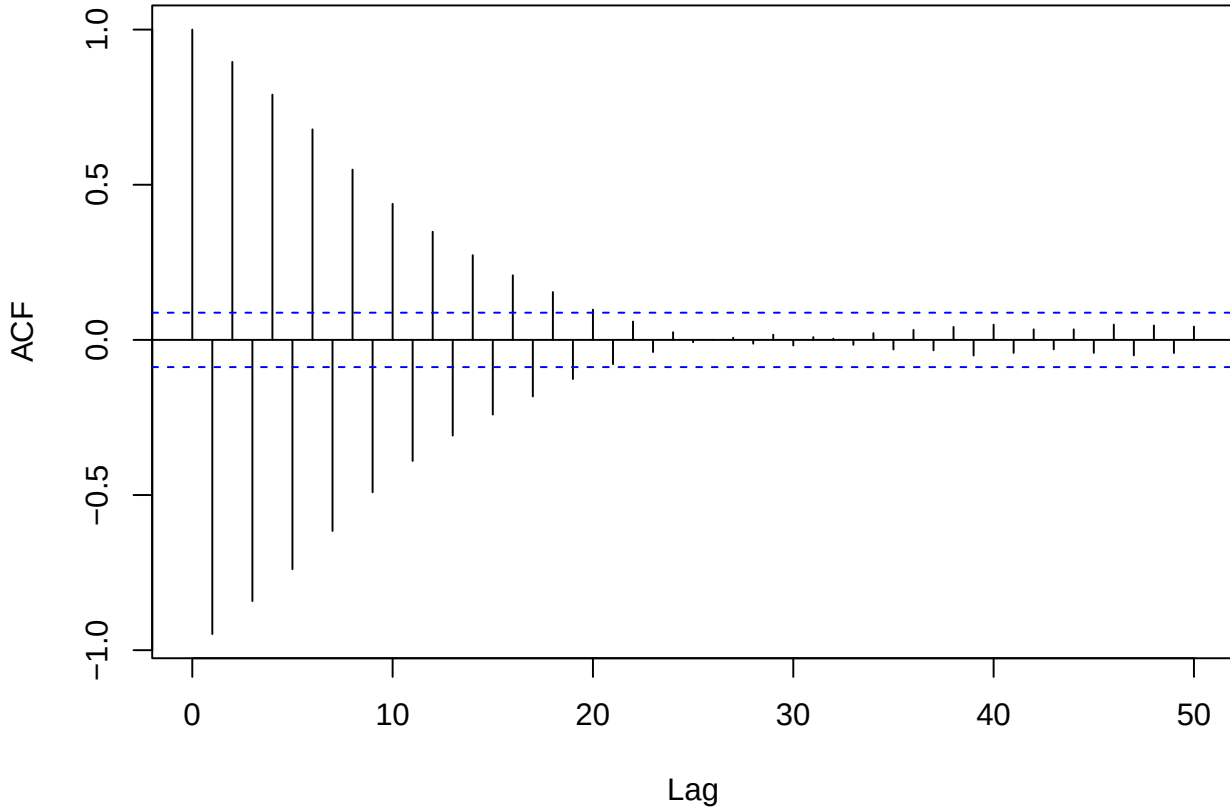
```

n<-500
X<-arima.sim(n,model=list(ar=covec[2:4]))
par(mar=c(4,4,1,2))
plot(X,type='l'); points(X,pch=19,cex=0.5)

```



```
acf(X, lag.max=50)
```



```
ARMAacf(ar=covec[2:4], lag.max=10)
```

```
##      0      1      2      3      4      5      6
## 1.000000 -0.9669932 0.9396628 -0.9115584 0.8844919 -0.8581854 0.8326682
##      7      8      9     10
## -0.8079084 0.7838850 -0.7605760 0.7379601
```

The process is stationary, and has ACF alternating in sign, with odd lags exhibiting negative correlation, and even lags exhibiting positive dependence, both decaying to negligible values before lag 25.

Invertibility: In the $ARMA(p, q)$ process in (1), the existence of a stationary and causal solution depends only on the AR polynomial $\Phi(z)$. However, it is sometimes useful to be able to re-write (1) in the form

$$\{\Theta(B)\}^{-1}\Phi(B)X_t = Z_t$$

filtering both sides of (1) by the inverse operator filter $\{\Theta(B)\}^{-1}$ to give the series expansion

$$Z_t = \pi(B)X_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} \quad (2)$$

where

$$\pi(B) = \{\Theta(B)\}^{-1}\Phi(B)$$

has the series expansion

$$\pi(B) = \sum_{j=0}^{\infty} \pi_j B^j.$$

If such a convergent expansion given in (2) exists, then $\{X_t\}$ is termed an *invertible* process. By identical arguments to those made for the AR model, the model is an invertible if

$$\sum_{j=0}^{\infty} |\pi_j| < \infty$$

and in the ARMA case, if the AR component is stationary and causal, the model is invertible if

$$\Theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q = \prod_{j=1}^q (1 - \omega_j z)$$

gives

$$\beta(z) = \{\Theta(z)\}^{-1} = \prod_{j=1}^q (1 - \omega_j z)^{-1}$$

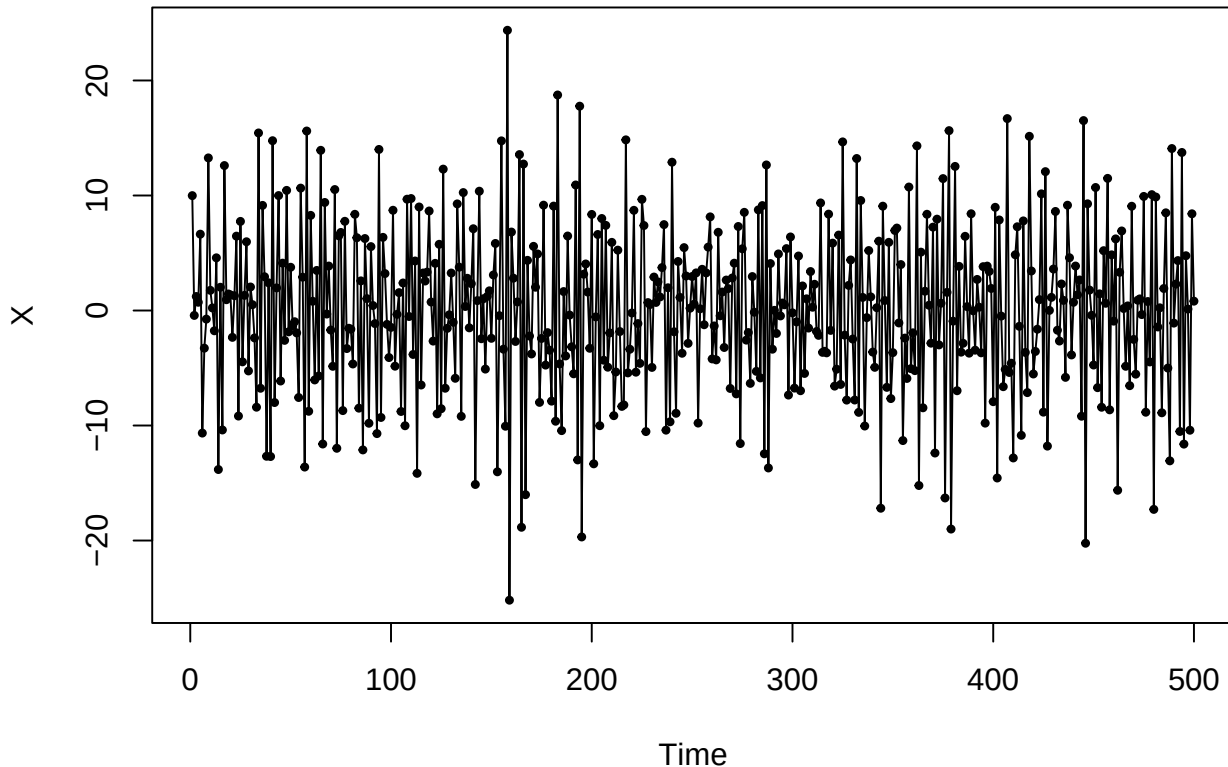
which has a convergent series expansion when $|z| = 1$. This is equivalent to requiring that $|\omega_j| < 1$ for all j .

Example: Consider the MA(2) process

$$X_t = Z_t - 5Z_{t-1} + 6Z_{t-2} = (1 - 2B)(1 - 3B)Z_t$$

This process is not invertible, as the expansions of $(1 - 2z)^{-1}$ and $(1 - 3z)^{-1}$ are divergent when $|z| = 1$. Despite this, the process is stationary, and well behaved. If we take $\{Z_t\}$ to be an IID Gaussian sequence with $\sigma_Z^2 = 1$, the process can be simulated.

```
Z<-rnorm(502)
X<-Z[3:502]-5*Z[2:501]+6*Z[1:500]
par(mar=c(4,4,2,2))
plot(X,type='l',xlab='Time')
points(X,pch=19,cex=0.5)
```



Under stationarity, with a causal AR component and an invertible MA component, we have that $\pi(B)$ has generating function

$$\pi(z) = \frac{\Phi(z)}{\Theta(z)} = \frac{\prod_{j=1}^p (1 - \xi_j z)}{\prod_{k=1}^q (1 - \omega_k z)} \quad (3)$$

whereas the linear process representation via $\Psi(B)$ with generating function

$$\Psi(z) = \frac{\Theta(z)}{\Phi(z)} = \frac{\prod_{k=1}^q (1 - \omega_k z)}{\prod_{j=1}^p (1 - \xi_j z)} \quad (4)$$

where $|\xi_j| < 1$ for all j and $|\omega_k| < 1$ for all k .

Non-causal case: In the AR(1) model, if $|\phi| > 1$, it is possible to find a stationary but non-causal solution to the AR equation: recall that if $|\phi| > 1$

$$X_t = \phi X_{t-1} + Z_t$$

with $\{Z_t\} \sim \text{WN}(0, \sigma_Z^2)$, then on rearrangement we have

$$X_{t-1} = \frac{1}{\phi} X_t - \frac{1}{\phi} Z_t \quad \text{or} \quad X_{t-1} = \phi_* X_t + Z_t^*$$

say, where $\phi_* = 1/\phi$ and where $Z_t^* = -Z_t/\phi$ so that $\{Z_t^*\} \sim \text{WN}(0, \sigma_Z^2/\phi^2)$. Using the standard recursion approach

$$\begin{aligned} X_t &= \phi_* X_{t+1} + Z_{t+1}^* = \phi_*^2 X_{t+2} + (\phi_* Z_{t+2}^* + Z_{t+1}^*) \\ &= \phi_*^3 X_{t+3} + (\phi_*^2 Z_{t+3}^* + \phi_* Z_{t+2}^* + Z_{t+1}^*) \end{aligned}$$

etc, which yields the solution

$$X_t = \sum_{j=0}^{\infty} \phi_*^j Z_{t+1+j}^* = - \sum_{j=0}^{\infty} \phi^{-(j+1)} Z_{t+1+j} \quad (5)$$

Example: Consider the AR(1) process

$$X_t = 1.5X_{t-1} + Z_t$$

This process is not stationary if we restrict attention to causal solutions. However, the stationary non-causal solution based on (5) can be computed. If we take $\{Z_t\}$ to be an IID Gaussian sequence with $\sigma_Z^2 = 1$, the process can be simulated. The variance of this process satisfies

$$\text{Var}[X_{t-1}] = \phi_*^2 \text{Var}[X_t] + \text{Var}[Z_t^*] \implies \text{Var}[X_t] = \frac{\text{Var}[Z_t^*]}{(1 - \phi_*^2)} = \frac{1/\phi_*^2}{(1 - \phi_*^2)} = \frac{1}{\phi^2 - 1}.$$

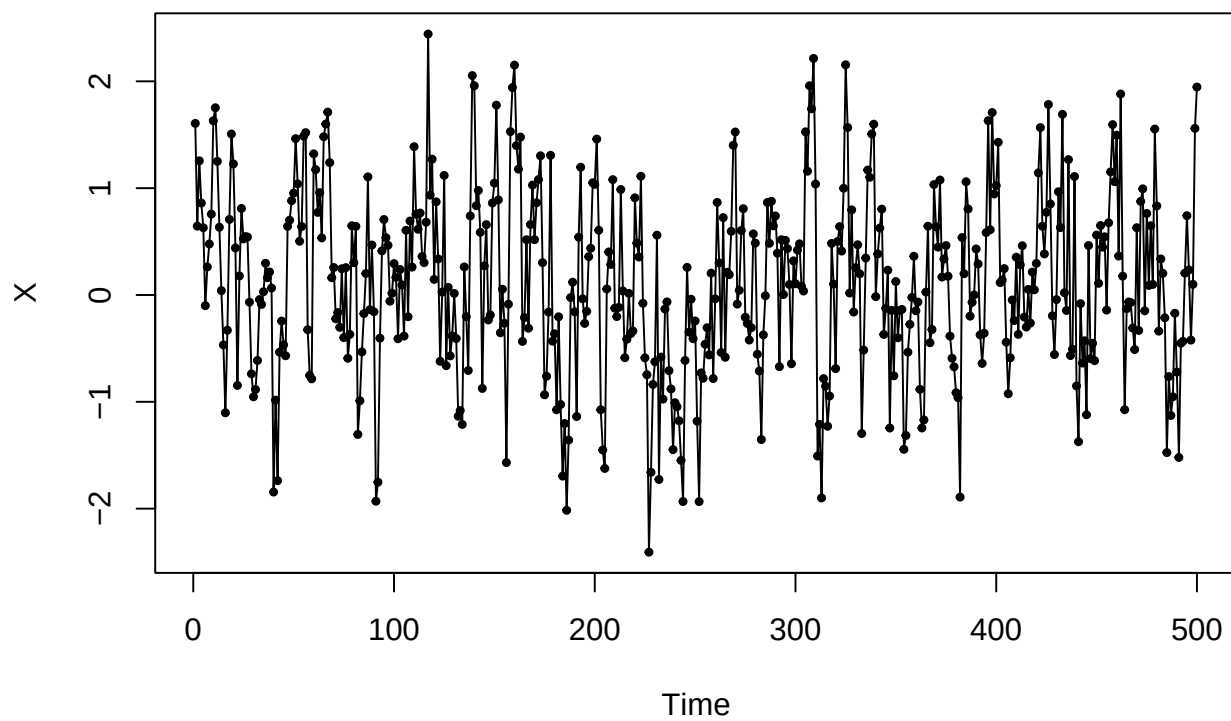
and as $\phi_* = 1/1.5 = 0.6667$, we have that $\text{Var}[X_t] = 0.8$.

```
phi<-1.5; phistar<-1/phi
Z<-rnorm(1001); Zstar<--Z/phi;
psivec<-phistar^c(0:500)
for(i in 1:500){
  X[i]<-sum(psivec*Zstar[(i+1):(i+1+500)]) #Generate X using (5)
}
var(X) #Variance
## [1] 0.7392994

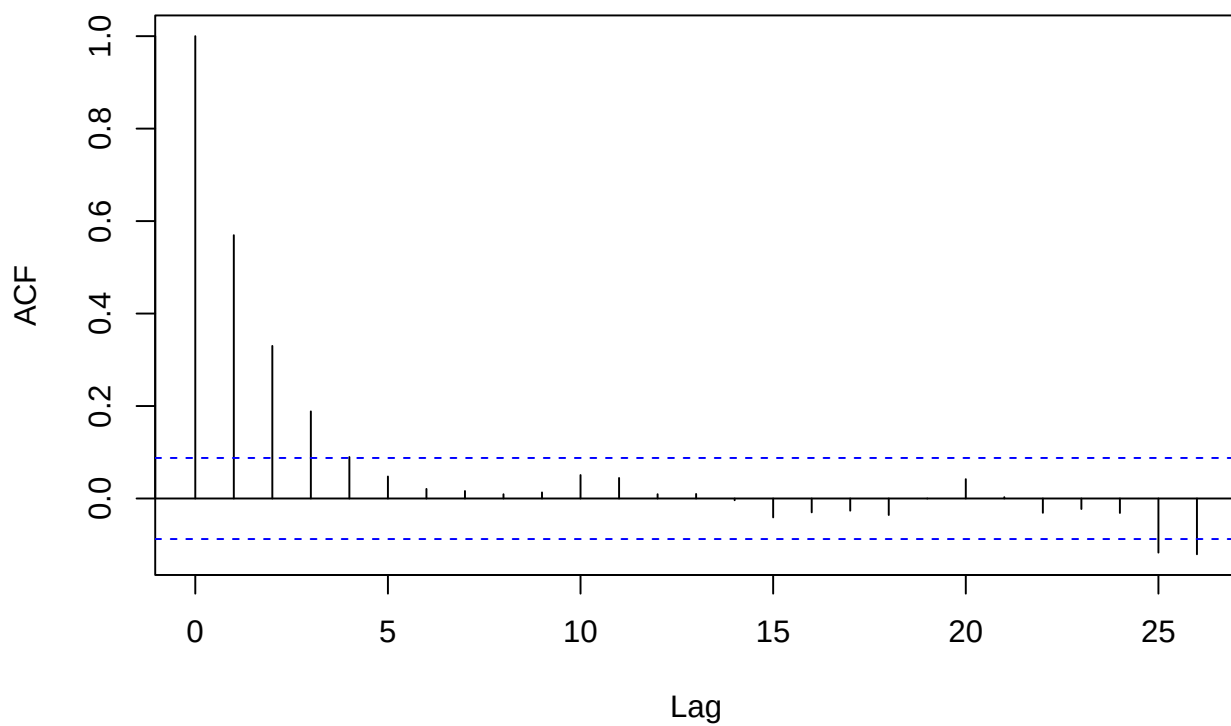
sdX<-sqrt(1/(phi^2-1))
X[500]<-rnorm(1,0,sdX) #Generate X using the recursion
for(i in 499:1){
  X[i]<-phistar*X[i+1]+Zstar[i+1]
}
var(X)
```

```
## [1] 0.7334067
```

```
par(mar=c(4,4,1,2))  
plot(X,type='l',xlab='Time')  
points(X,pch=19,cex=0.5)
```



```
acf(X)
```



We can compute the ACVF for $h \geq 0$ as

$$\begin{aligned}\mathbb{E}[X_t X_{t+h}] &= \mathbb{E} \left[\left(\sum_{j=0}^{\infty} \phi_*^j Z_{t+1+j}^* \right) \left(\sum_{k=0}^{\infty} \phi_*^k Z_{t+h+1+k}^* \right) \right] \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \phi_*^{j+k} \mathbb{E} [Z_{t+1+j}^* Z_{t+h+1+k}^*]\end{aligned}$$

The expectation is only non-zero when the indices match and $(t+1+j) = (t+h+1+k)$, that is, when $k = j - h$, in which case the expectation is $\text{Var}[Z_t^*] = \sigma_Z^2 / \phi^2$. Therefore

$$\mathbb{E}[X_t X_{t+h}] = \frac{\sigma_Z^2}{\phi^2} \sum_{j=h}^{\infty} \phi_*^{2j-h} = \frac{\sigma_Z^2}{\phi^2} \sum_{j=0}^{\infty} \phi_*^{2(j+h)-h} = \frac{\sigma_Z^2}{\phi^2} \phi_*^h \sum_{j=0}^{\infty} \phi_*^{2j} = \frac{\sigma_Z^2}{\phi^2} \phi_*^h \frac{1}{1 - \phi_*^2}.$$

as $\phi_* = 1/\phi$, after summing the geometric progression. Simplifying, we obtain for the stationary but non-causal solution that

$$\gamma_X(h) = \frac{\sigma_Z^2}{\phi^2 - 1} \frac{1}{\phi^h} \quad h = 0, 1, 2, \dots$$

```
h<-1:10
gamvec<-1/(phi^2-1)*(1/phi)^h
gamvec

## [1] 0.53333333 0.35555556 0.23703704 0.15802469 0.10534979 0.07023320
## [7] 0.04682213 0.03121475 0.02080984 0.01387322

rhovec<-gamvec
rhovec/rhovec[1]

## [1] 1.00000000 0.66666667 0.44444444 0.29629630 0.19753086 0.13168724
## [7] 0.08779150 0.05852766 0.03901844 0.02601229

acf(X,lag.max=10,plot=F)

##
## Autocorrelations of series 'X', by lag
##
## 0 1 2 3 4 5 6 7 8 9 10
## 1.000 0.570 0.330 0.188 0.090 0.048 0.021 0.016 0.009 0.013 0.051
```