

MATH 545: INTRODUCTION TO TIME SERIES ANALYSIS

WINTER 2025

RECAP ON COMPLEX NUMBERS AND R FUNCTIONALITY

A complex number z takes the form

$$z = x + yi$$

where x and y are real numbers and $i = \sqrt{-1}$.

- x is the Real part of z , $\text{Re}(z)$, y is the Imaginary part of z , $\text{Im}(z)$.
- The Modulus of z , $|z|$, is defined by

$$|z| = \sqrt{x^2 + y^2}$$

- The Argument of z , $\arg(z)$, is defined by

$$\arg(z) = \arctan(y/x)$$

with range $(-\pi, \pi]$.

- We have that

$$z = |z|e^{i\arg(z)}$$

For example, consider $z = 2 + 3i$.

```
z<-2+3i
z
## [1] 2+3i

Re(z)    #Real part
## [1] 2

Im(z)    #Imaginary part
## [1] 3

Mod(z)    #Modulus
## [1] 3.605551

Mod(z)^2 #Square of modulus
## [1] 13

Arg(z)    #Argument
## [1] 0.9827937

atan(3/2)
## [1] 0.9827937

Mod(z)*exp(1i*Arg(z))
## [1] 2+3i
```

Complex numbers can be manipulated using standard algebra: if $z_1 = x_1 + y_1i$ and $z_2 = x_2 + y_2i$

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i = |z_1| |z_2| e^{i(\arg(z_1) + \arg(z_2))}$$

$$z_1^{-1} = \frac{x_1}{x_1^2 + y_1^2} - \frac{y_1}{x_1^2 + y_1^2}i = |z_1|^{-1} e^{-i \arg(z_1)}$$

Polynomials: Consider the polynomial

$$A(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{p-1} z^{p-1} + a_p z^p$$

where $a_0, a_1, \dots, a_p$ are real values. Roots of the polynomial are values of z for which $A(z) = 0$. When discussing the roots, without loss of generality we may set $a_0 = 1$ or $a_p = 1$. The usual approach takes $a_p = 1$, in which case we may write

$$A(z) = \prod_{j=1}^p (z - z_j) \quad (1)$$

and we term $z_1, z_2, \dots, z_p$ the roots of the polynomial. The roots may be

- entirely real, with no imaginary part
- complex, in which case the roots must appear in complex conjugate pairs, say

$$z_j = x + yi \quad z_{j+1} = x - yi = \bar{z}_j.$$

If instead we take $a_0 = 1$, then the factorization in (1) can also be expressed

$$A(z) = \prod_{j=1}^p (1 - r_j z) \quad (2)$$

and the values $r_1, \dots, r_p$ that make $A(z) = 0$ are termed the *reciprocal roots*, as $r_j = 1/z_j$. In R, the function `polyroot` can be used to find the z_j and the r_j .

EXAMPLE: Suppose $p = 3$, and

$$A(z) = (z - 1)(z - 2i)(z + 2i) = (z - 1)(z^2 + 4) = z^3 - z^2 + 4z - 4 = -4 + 4z - z^2 + z^3.$$

so $(a_0, a_1, a_2, a_3) = (-4, 4, -1, 1)$.

```
avec<-c(-4,4,-1,1)
#polyroot takes the arguments in increasing order of z^j
polyroot(avec)

## [1] 1.000000e+00-4.230797e-19i 1.057699e-19+2.000000e+00i
## [3] -1.057699e-19-2.000000e+00i

round(polyroot(avec),10) #Roots

## [1] 1+0i 0+2i 0-2i

round(1/polyroot(avec),10) #Reciprocal roots

## [1] 1+0.0i 0-0.5i 0+0.5i
```

The negligible real and imaginary parts arise due to the numerical method of solution. Note that if $r_1 = 1$, $r_2 = -i/2$ and $r_3 = i/2$

$$(1 - z)(1 + (i/2)z)(1 - (i/2)z) = (1 - z)(1 + z^2/4) = (1 - z + z^2/4 - z^3/4) = -\frac{1}{4}(-4 + 4z - z^2 + z^3)$$

Note, however that an alternative calculation method works to compute the reciprocal roots directly. Consider, for $z \neq 0$, the polynomial

$$A^*(z) = A(1/z) = -4 + \frac{4}{z} - \frac{1}{z^2} + \frac{1}{z^3} = \frac{1}{z^3}(-4z^3 + 4z^2 - z + 1).$$

We have that

$$A(1/z) = (1/z - z_1)(1/z - z_2)(1/z - z_3) = \frac{1}{z^3}(1 - z_1z)(1 - z_2z)(1 - z_3z)$$

so the solutions to

$$A^*(z) = A(1/z) = 0$$

are precisely the reciprocal roots. Therefore, if we reverse the order of the arguments before passing them into the function, we obtain the reciprocal roots directly from `polyroot`.

```
avec<-rev(c(-4,4,-1,1))
avec                                     #Reverse order coefficients

## [1] 1 -1 4 -4

polyroot(avec)

## [1] 5.562005e-18+0.5i 1.185603e-17-0.5i 1.000000e+00+0.0i

round(polyroot(avec),10)               #Reciprocal roots

## [1] 0+0.5i 0-0.5i 1+0.0i

round(1/polyroot(avec),10)             #Roots

## [1] 0-2i 0+2i 1+0i
```