

Chapter 6

Spectral Methods

6.1 Spectral Analysis of Time Series

The **spectral** or **frequency domain** representation of time series $\{X_t\}$ focusses on stationary processes, and constructs an alternative view to the time domain representation studied previously.

The focus on time domain methods is paramount in statistics because observed data are typically collected **sequentially** in time.

For stationary processes, the non-negative definite autocovariance function admits a **spectral** decomposition which facilitates alternative inference approaches.

6.2 Spectral Representation

The ACVF of a (zero-mean) stationary process $\{X_t\}$

$$\gamma_X(h) = \mathbb{E}[X_t X_{t+h}]$$

is a symmetric, non-negative definite function of the discrete argument

$$h = 0, \pm 1, \pm 2, \dots$$

We exploit the fact that $\gamma_X(h)$ has a unique representation in terms of **frequency contributions**.

The Spectral Density Function

Let $\{X_t\}$ be stationary and zero-mean, with ACVF $\gamma_X(\cdot)$, with

$$\sum_{h=-\infty}^{\infty} |\gamma_X(h)| < \infty.$$

Then the **spectral density function** (SDF) of $\{X_t\}$ is the function $f_X(\cdot)$ defined by

$$f_X(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\lambda h} \gamma_X(h) \quad \lambda \in \mathbb{R}$$

The Spectral Density Function

Note that $f_X(\lambda)$ is well-defined, as $|e^{-i\lambda h}| = 1$, so

$$\sum_{h=-\infty}^{\infty} |e^{-i\lambda h} \gamma_X(h)| \leq \sum_{h=-\infty}^{\infty} |e^{-i\lambda h}| |\gamma_X(h)| = \sum_{h=-\infty}^{\infty} |\gamma_X(h)| < \infty.$$

and $f_X(\lambda)$ is **periodic** as

$$e^{-i\lambda h} = \cos(\lambda h) - i \sin(\lambda h)$$

so we can restrict attention to $\lambda \in (-\pi, \pi]$

The Spectral Density Function

Example: AR(1)

Consider the stationary AR(1) model

$$(1 - \phi B)X_t = Z_t$$

so that for $h \geq 0$

$$\gamma_X(h) = \frac{\sigma_Z^2}{1 - \phi^2} \phi^h$$

The Spectral Density Function

Example: AR(1)

Then

$$\begin{aligned}f_X(\lambda) &= \frac{\sigma_Z^2}{1 - \phi^2} \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\lambda h} \phi^{|h|} \\&= \frac{\sigma_Z^2}{2\pi(1 - \phi^2)} \left[\frac{1}{1 - \phi e^{-i\lambda}} + \sum_{h=-\infty}^{-1} e^{-i\lambda h} \phi^{-h} \right] \\&= \frac{\sigma_Z^2}{2\pi(1 - \phi^2)} \left[\frac{1}{1 - \phi e^{-i\lambda}} + \sum_{h=1}^{\infty} e^{i\lambda h} \phi^h \right] \\&= \frac{\sigma_Z^2}{2\pi(1 - \phi^2)} \left[\frac{1}{1 - \phi e^{-i\lambda}} + \frac{1}{1 - \phi e^{i\lambda}} - 1 \right]\end{aligned}$$

The Spectral Density Function

Example: AR(1)

Thus

$$\begin{aligned} f_X(\lambda) &= \frac{\sigma_Z^2}{2\pi(1-\phi^2)} \frac{2 - \phi e^{-i\lambda} - \phi e^{i\lambda} - (1 - \phi e^{-i\lambda})(1 - \phi e^{i\lambda})}{(1 - \phi e^{-i\lambda})(1 - \phi e^{i\lambda})} \\ &= \frac{\sigma_Z^2}{2\pi} \frac{1}{(1 - \phi e^{-i\lambda})(1 - \phi e^{i\lambda})} \\ &= \frac{\sigma_Z^2}{2\pi} \frac{1}{|1 - \phi e^{-i\lambda}|^2} \end{aligned}$$

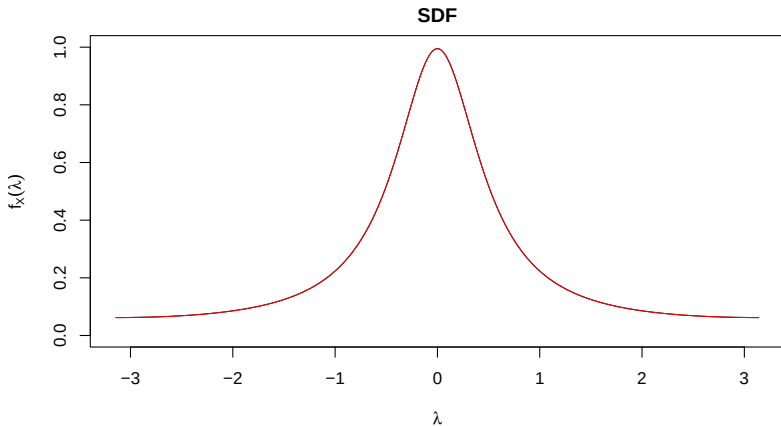
The Spectral Density Function

```
phi<-0.6
hvec<-0:30
acvf.vals<-1/(1-phi^2)*phi^hvec
lamvec<-seq(-pi,pi,by=0.001)
sdf.vec<-lamvec*0
hvec<-0:30
for(j in 1:length(lamvec)){
  sdf.vec[j]<-sum(exp(-1i*lamvec[j]*hvec[-1])*acvf.vals[2:31])+acvf.vals[1]+
    sum(exp(1i*lamvec[j]*hvec[-1])*acvf.vals[2:31])
}
sdf.vec<-sdf.vec/(2*pi)

exact.sdf<-1/(2*pi)*(1/Mod(1-phi*exp(-1i*lamvec))^2)
```

The Spectral Density Function

```
par(mar=c(4,4,2,2));plot(lamvec,sdf.vec,type='l',xlab=expression(lambda),  
    ylab=expression(f[X](lambda)), ylim=range(0,1),main = 'SDF');lines(lamvec,exact.sdf,col='red')
```



The Spectral Density Function

Example: ARMA(1,1)

Consider the stationary ARMA(1, 1) model

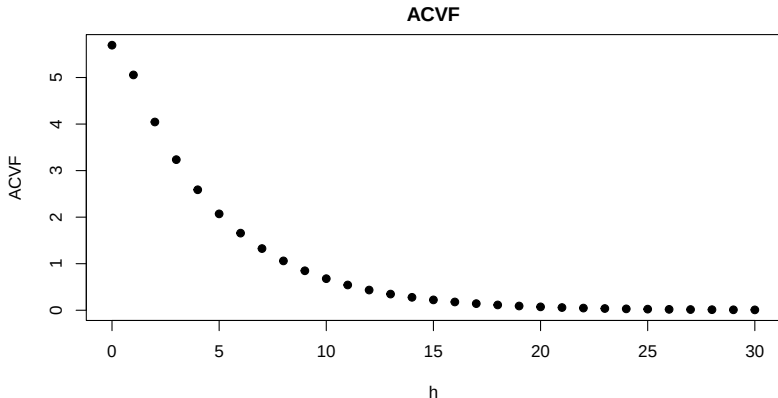
$$(1 - 0.8B)X_t = (1 + 0.5B)Z_t$$

We may inspect

- the ACVF
- the SDF

The Spectral Density Function

```
library(ltsa)
acvf.vals<-tacvfARMA(phi = 0.8, theta = -0.5, maxLag = 30)
par(mar=c(4,4,2,2));plot(0:30,acvf.vals,xlab='h', ylab='ACVF', main = 'ACVF',pch=19)
```

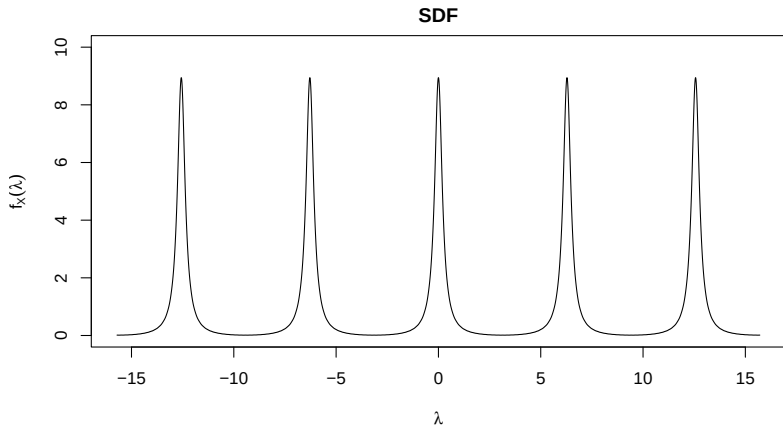


The Spectral Density Function

```
lamvec<-seq(-5*pi,5*pi,by=0.001)
sdf.vec<-lamvec*0
hvec<-0:30
for(j in 1:length(lamvec)){
  sdf.vec[j]<-sum(exp(-1i*lamvec[j]*hvec[-1])*acvf.vals[2:31])+acvf.vals[1]+
    sum(exp(1i*lamvec[j]*hvec[-1])*acvf.vals[2:31])
}
sdf.vec<-sdf.vec/(2*pi)
```

The Spectral Density Function

```
par(mar=c(4,4,2,2));plot(lamvec,sdf.vec,type='l',xlab=expression(lambda),  
ylab=expression(f[X](lambda)), ylim=range(0,10),main = 'SDF')
```



The Spectral Density Function

Example: ARMA(1,1)

It can be shown that

$$f_X(\lambda) = \frac{\sigma_Z^2}{2\pi} \left| \frac{1 + 0.5e^{-i\lambda}}{1 - 0.8e^{-i\lambda}} \right|^2$$

We plot on the range

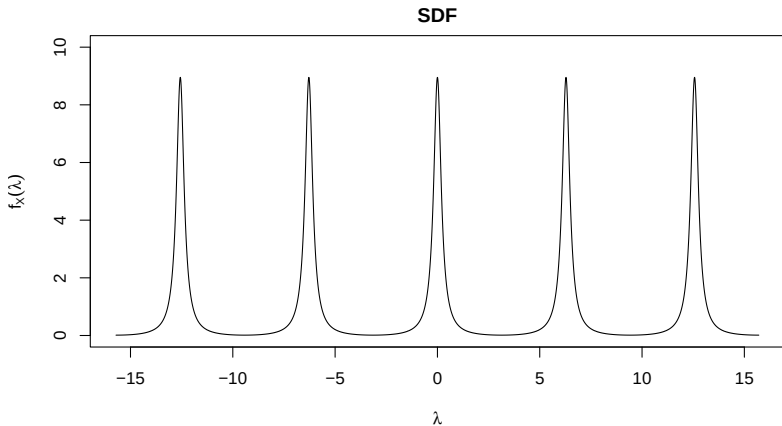
- $[-5\pi, 5\pi]$
- $[-\pi, \pi]$
- $[0, \pi]$

The Spectral Density Function

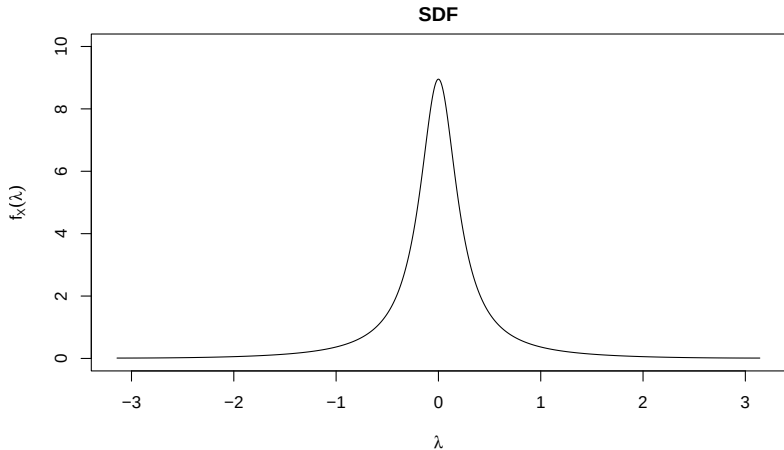
```
lamvec<-seq(-5*pi,5*pi,by=0.001)
sdf.vec<-(1/(2*pi))*Mod((1+0.5*exp(-1i*lamvec))/(1-0.8*exp(-1i*lamvec)))^2
```

The Spectral Density Function

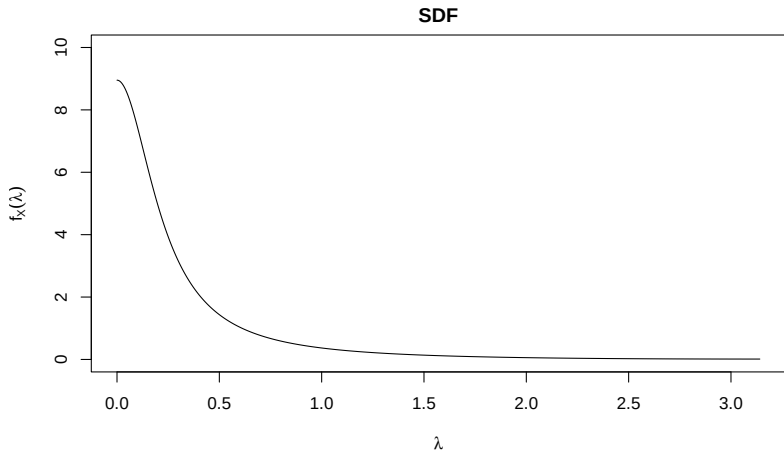
```
par(mar=c(4,4,2,2));plot(lamvec,sdf.vec,type='l',xlab=expression(lambda),  
ylab=expression(f[X](lambda)), ylim=range(0,10),main = 'SDF')
```



The Spectral Density Function

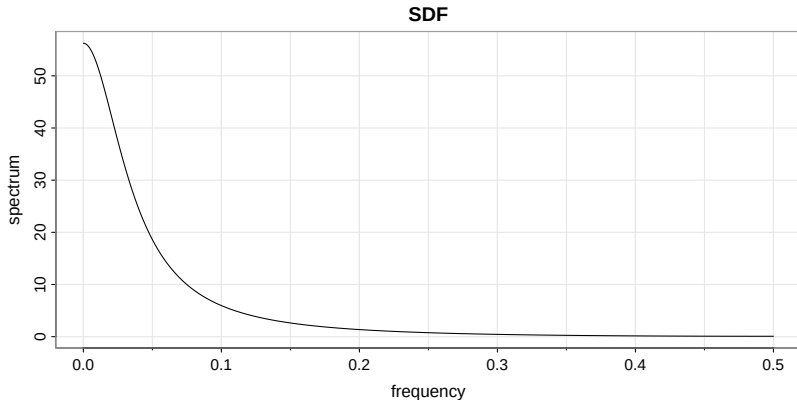


The Spectral Density Function



The Spectral Density Function

```
library(astsa)
par(mar=c(4,4,2,2))
arma.spec(ar = 0.8, ma = 0.5, var.noise = 1, n.freq = 500, main='SDF', frequency=1, plot=TRUE)
```



The Spectral Density Function

This computes the scaled version in terms of φ , where

$$\varphi = \frac{\lambda}{2\pi} \quad \text{so that} \quad -\frac{1}{2} < \varphi \leq \frac{1}{2}$$

in which case

$$f_X^*(\varphi) = \sum_{h=-\infty}^{\infty} e^{-i2\pi\varphi h} \gamma_X(h)$$

or more explicitly

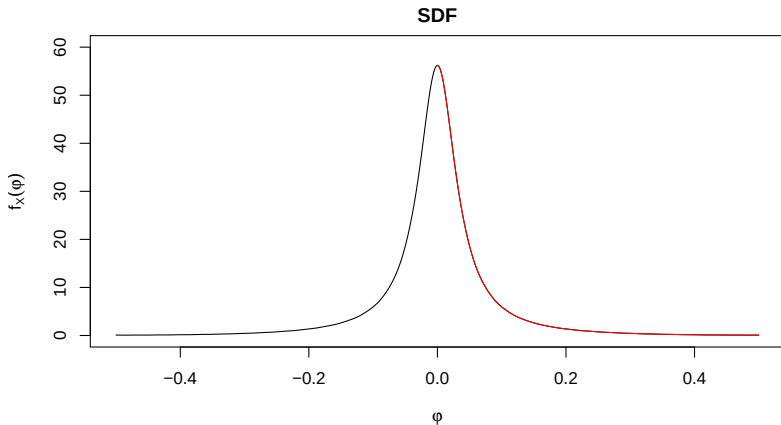
$$f_X^*(\varphi) = \sigma_Z^2 \left| \frac{1 + 0.5e^{-i2\pi\varphi}}{1 - 0.8e^{-i2\pi\varphi}} \right|^2$$

The Spectral Density Function

```
vvec0<-seq(-0.5,0.5,by=0.001)
sdf.vec0<-vvec0*0
hvec<-0:30
for(j in 1:length(vvec0)){
  sdf.vec0[j]<-sum(exp(-1i*2*pi*vvec0[j]*hvec[-1])*acvf.vals[2:31])+acvf.vals[1]+
    sum(exp(1i*2*pi*vvec0[j]*hvec[-1])*acvf.vals[2:31])
}
vvec1<-seq(0,0.5,by=0.001)
sdf.vec1<-Mod((1+0.5*exp(-1i*2*pi*vvec1))/(1-0.8*exp(-1i*2*pi*vvec1)))^2
```

The Spectral Density Function

```
par(mar=c(4,4,2,2));plot(vvec0,sdf.vec0,type='l',xlab=expression(varphi),  
ylab=expression(f[X](varphi)), ylim=range(0,60),main = 'SDF'); lines(vvec1,sdf.vec1,col='red')
```



The Spectral Density Function

We have that

(i) $f_X(\lambda) = f_X(-\lambda)$ and

$$\begin{aligned}f_X(\lambda) &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\lambda h} \gamma_X(h) \\&= \frac{1}{2\pi} \left[\gamma_X(0) + \sum_{h=1}^{\infty} (e^{-i\lambda h} + e^{i\lambda h}) \gamma_X(h) \right] \quad \gamma(-h) = \gamma(h) \\&= \frac{1}{2\pi} \left[\gamma_X(0) + \sum_{h=1}^{\infty} 2 \cos(\lambda h) \gamma_X(h) \right] \\&= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \cos(\lambda h) \gamma_X(h) \quad \text{as } \cos(-\lambda h) = \cos(\lambda h)\end{aligned}$$

(ii) $f_X(\lambda) \geq 0$, $-\pi < \lambda \leq \pi$.

The Spectral Density Function

(iii) Inversion formula:

$$\begin{aligned}\int_{-\pi}^{\pi} e^{ih\lambda} f_X(\lambda) d\lambda &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \sum_{k=-\infty}^{\infty} e^{i\lambda(h-k)} \gamma_X(k) \right\} d\lambda \\ &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_X(k) \left\{ \int_{-\pi}^{\pi} e^{i\lambda(h-k)} d\lambda \right\} \\ &= \gamma_X(h).\end{aligned}$$

The Spectral Density Function

The last result follows as if $k \neq h$, setting $j = h - k$

$$\begin{aligned}\int_{-\pi}^{\pi} e^{i\lambda(h-k)} d\lambda &= \int_{-\pi}^{\pi} [\cos(\lambda j) + i \sin(\lambda j)] d\lambda \\ &= \int_{-\pi}^{\pi} \cos(\lambda j) d\lambda + i \int_{-\pi}^{\pi} \sin(\lambda j) d\lambda \\ &= 0.\end{aligned}$$

The Spectral Density Function

Note: The summability condition

$$\sum_{h=-\infty}^{\infty} |\gamma_X(h)| < \infty$$

is not a necessary condition for the existence of a SDF.

The Spectral Density Function

There is a 1-1 correspondence between ACVFs and spectral density functions; if

(i) $f_X(\lambda) \geq 0$, for all $\lambda \in (0, \pi]$.

(ii) For $h = 0, \pm 1, \pm 2, \dots$, if

$$\gamma_X(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f_X(\lambda) d\lambda$$

then f_X is the SDF of $\{X_t\}$ with ACVF $\gamma_X(h)$. Note that

$$\gamma_X(h) = \int_{-\pi}^{\pi} [\cos(h\lambda) + i \sin(h\lambda)] f_X(\lambda) d\lambda = \int_{-\pi}^{\pi} \cos(h\lambda) f_X(\lambda) d\lambda.$$

The Spectral Density Function

A real-valued function f defined on $(-\pi, \pi]$ is the SDF of a stationary process if and only if

- (i) f is even: $f(\lambda) = f(-\lambda)$ for all λ .
- (ii) $f(\lambda) > 0$
- (iii)

$$\int_{-\pi}^{\pi} f(\lambda) d\lambda < \infty$$

The Spectral Density Function

Suppose f satisfies (i)-(iii), then

$$\gamma_X(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f_X(\lambda) d\lambda$$

is even. Let $\mathbf{a} \in \mathbb{R}^n$. Then

$$\begin{aligned}\mathbf{a}^\top \mathbf{\Gamma} \mathbf{a} &= \sum_{r=1}^n \sum_{s=1}^n a_r \gamma_X(r-s) a_s = \int_{-\pi}^{\pi} \sum_{r=1}^n \sum_{s=1}^n a_r a_s e^{i\lambda(r-s)} f(\lambda) d\lambda \\ &= \int_{-\pi}^{\pi} \left| \sum_{r=1}^n a_r e^{i\lambda r} \right|^2 f(\lambda) d\lambda \geq 0\end{aligned}$$

so $\gamma_X(\cdot)$ is non-negative definite, and hence the ACVF for a stationary process.

The Spectral Density Function

Note:

- f_X is the **Fourier transform** of γ_X
- γ_X is the **inverse Fourier transform** of f_X .

The properties of the SDF f_X are similar to a pdf, with the additional condition that f_X must be **symmetric about zero**, and with the integrability relaxed to

$$\int_{-\pi}^{\pi} f_X(\lambda) d\lambda = C < \infty$$

rather than having $C = 1$.

Spectral Representation Theorem

Theorem: Uniqueness

A function $\gamma(\cdot)$, $\gamma : \mathbb{Z} \longrightarrow \mathbb{R}$, is an autocovariance function if and only if there exists a right continuous, non-decreasing bounded function F on $[-\pi, \pi]$ with $F(-\pi) = 0$, and for all $h \in \mathbb{Z}$

$$\gamma(h) = \int_{(-\pi, \pi]} e^{ih\lambda} dF(\lambda).$$

As the corresponding series $\{\gamma(h)\}$ is real-valued, F satisfies

$$\int_{(a, b]} dF(x) = \int_{(-b, -a]} dF(x)$$

Spectral Representation Theorem

F is a **generalized distribution function** as $F(\lambda)/F(\pi)$ is a distribution function on $[-\pi, \pi]$.

F is termed the **spectral distribution function**; it can be purely discrete, absolutely continuous, or mixed. If F is absolutely continuous

$$F(\lambda) = \int_{-\pi}^{\lambda} f(x) \, dx \quad \lambda \in (-\pi, \pi]$$

then f is the **spectral density function**.

Example

Example: Harmonic process

Suppose

$$X_t = U \cos(\omega t) + V \sin(\omega t)$$

where U and V are uncorrelated, with

$$\mathbb{E}[U] = \mathbb{E}[V] = 0 \qquad \text{Var}[U] = \text{Var}[V] = 1$$

Then by a previous result

$$\gamma_X(h) = \cos(\omega h) \qquad h \in \mathbb{Z}$$

Example

Example: Harmonic process

Now

$$\sum_{h=-\infty}^{\infty} |\gamma_X(h)| = \sum_{h=-\infty}^{\infty} |\cos(\omega h)| = \infty$$

However, let

$$F_X(\lambda) = \begin{cases} 0 & \lambda < -\omega \\ \frac{1}{2} & -\omega \leq \lambda < \omega \\ 1 & \omega \leq \lambda \end{cases}$$

Example

Example: Harmonic process

Then

$$f_X(\lambda) = \begin{cases} \frac{1}{2} & \lambda = -\omega \\ \frac{1}{2} & \lambda = \omega \\ 0 & \text{otherwise} \end{cases}$$

and

$$\int_{(-\pi, \pi)} e^{ih\lambda} dF_X(\lambda) = \frac{1}{2}e^{-ih\omega} + \frac{1}{2}e^{ih\omega} = \cos(h\omega)$$

Example: Extension

Example: Superposition of harmonic processes

Suppose

$$X_t = \sum_{j=1}^k [U_j \cos(\omega_j t) + V_j \sin(\omega_j t)]$$

where $U_1, \dots, U_k$ and $V_1, \dots, V_k$ are uncorrelated, with

$$\mathbb{E}[U_j] = \mathbb{E}[V_j] = 0 \qquad \text{Var}[U_j] = \text{Var}[V_j] = \sigma_j^2$$

and where $0 < \omega_1 < \dots < \omega_k < \pi$. Then

$$\gamma_X(h) = \sum_{j=1}^k \sigma_j^2 \cos(\omega_j h) \qquad h \in \mathbb{Z}.$$

Example: Extension

Example: Superposition of harmonic processes

Then

$$F_X(\lambda) = \sum_{j=1}^k \sigma_j^2 F_j(\lambda)$$

where

$$F_j(\lambda) = \begin{cases} 0 & \lambda < -\omega_j \\ \frac{1}{2} & -\omega_j \leq \lambda < \omega_j \\ 1 & \omega_j \leq \lambda \end{cases}$$

Note that

$$\text{Var}[X_t] = \gamma_X(0) = \sum_{j=1}^k \sigma_j^2$$

Spectral decomposition

This result shows that the variance of $\{X_t\}$ can be decomposed into the variances contributed by the harmonic components at different frequencies

$$\omega_1, \dots, \omega_k$$

In the limit as $k \rightarrow \infty$

$$X_t \stackrel{\text{m.s.}}{=} \int_{(-\pi, \pi]} e^{it\lambda} dZ(\lambda)$$

where $\{Z(\lambda) : -\pi < \lambda \leq \pi\}$ is a non-decreasing stochastic process.

In general, Z is a **complex-valued** process which has uncorrelated increments.

Spectral decomposition

A complex-valued process, $\{X_t\}$, is a stochastic process with complex-valued elements

$$X_t = X_{1,t} + iX_{2,t}$$

say, where

$$\mathbb{E}[X_t] = \mathbb{E}[X_{1,t}] + i\mathbb{E}[X_{2,t}]$$

$$\begin{aligned}\mathbb{E}[X_t X_{t+h}] &= \mathbb{E}[(X_{1,t} + iX_{2,t})(X_{1,t+h} + iX_{2,t+h})] \\ &= \mathbb{E}[X_{1,t}X_{1,t+h} - X_{2,t}X_{2,t+h}] \\ &\quad + i\mathbb{E}[X_{1,t}X_{2,t+h} + X_{2,t}X_{1,t+h}]\end{aligned}$$

etc.

Spectral decomposition

We have

$$\begin{aligned} X_t &= \int_{(-\pi, \pi]} e^{it\lambda} dZ(\lambda) \\ &= \int_{(-\pi, \pi]} [\cos(t\lambda) + i \sin(t\lambda)] dZ(\lambda) \end{aligned}$$

where $Z(\lambda)$ is a random process on $(-\pi, \pi]$. By symmetry, we need only consider $Z(\lambda)$ for $0 \leq \lambda \leq \pi$.

Spectral decomposition

For finite k ,

$$dZ(\lambda) = \begin{cases} \frac{U_j + iV_j}{2} & \lambda = -\omega_j, j = 1, \dots, k \\ \frac{U_j - iV_j}{2} & \lambda = \omega_j, j = 1, \dots, k \\ 0 & \text{otherwise} \end{cases}$$

Here

$$\mathbb{E}[dZ(\lambda)\overline{dZ(\lambda)}] = \mathbb{E}[(U_j^2 + V_j^2)/4] = \frac{\sigma_j^2}{2}$$

for $\lambda = \pm\omega_j, j = 1, \dots, k$ and zero otherwise.

Example: White noise process

Example: $Z_t \sim \text{WN}(0, \sigma_Z^2)$

If $Z_t \sim \text{WN}(0, \sigma_Z^2)$, then $\gamma_Z(0) = \sigma_Z^2$, $\gamma_Z(h) = 0$ for all $h \neq 0$.
Then, for $-\pi < \lambda \leq \pi$,

$$\begin{aligned} f_Z(\lambda) &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma_Z(h) \\ &= \frac{1}{2\pi} [\cos(0) - i \sin(0)] \sigma_Z^2 \\ &= \frac{\sigma_Z^2}{2\pi} \end{aligned}$$

Spectral decomposition

λ is an **angular frequency**:

- For a time series with dominant **high frequency** components, $f_X(\lambda)$ is large when λ is near π , and the series exhibits **rapid variation**.
- For a time series with dominant **low frequency** components, $f_X(\lambda)$ is large when λ is near zero, and the series exhibits **slow variation**.

Spectral decomposition

Often we use the scaled frequency φ ,

$$\varphi = \frac{\lambda}{2\pi} \quad \text{so that} \quad -\frac{1}{2} < \varphi \leq \frac{1}{2}$$

so that

$$f_X^*(\varphi) = \sum_{h=-\infty}^{\infty} e^{-i2\pi\varphi h} \gamma_X(h)$$

and

$$\gamma_X(h) = \int_{-1/2}^{1/2} e^{i2\pi\varphi h} f_X^*(\varphi) d\varphi$$

6.3 Representation of ARMA processes

Suppose $\Psi \equiv \{\psi_j : j = 0, \pm 1, \pm 2, \dots\}$ is a linear filter with

$$\sum_{j=-\infty}^{\infty} |\psi_j| < \infty.$$

and suppose $\{X_t\}$ is zero mean with SDF $f_X(\cdot)$.

Let

$$\Psi(z) = \sum_{j=-\infty}^{\infty} \psi_j z^j \quad z \in \mathbb{C}.$$

and note that

$$\overline{\Psi(z)} = \Psi(\bar{z}).$$

6.3 Representation of ARMA processes

If

$$Y_t = \sum_{j=-\infty}^{\infty} \psi_j X_{t-j} = \Psi(B)X_t.$$

then $\{Y_t\}$ is stationary and zero mean, with

$$\gamma_Y(h) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_X(h + k - j)$$

by the previous result for ACVFs

6.3 Representation of ARMA processes

Then

$$\begin{aligned}f_Y(\lambda) &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\lambda h} \gamma_Y(h) \\&= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\lambda h} \left\{ \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_X(h + k - j) \right\} \\&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \left\{ \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\lambda h} \gamma_X(h + k - j) \right\} \\&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \left\{ \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} e^{-i\lambda(l+j-k)} \gamma_X(l) \right\}\end{aligned}$$

setting $l = h + k - j$.

6.3 Representation of ARMA processes

Therefore

$$\begin{aligned}f_Y(\lambda) &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k e^{-i\lambda(j-k)} \left\{ \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} e^{-i\lambda l} \gamma_X(l) \right\} \\&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k e^{-i\lambda(j-k)} f_X(\lambda) \\&= \left(\sum_{j=-\infty}^{\infty} \psi_j e^{-i\lambda j} \right) \left(\sum_{k=-\infty}^{\infty} \psi_k e^{i\lambda k} \right) f_X(\lambda) \\&= \Psi(e^{-i\lambda}) \Psi(e^{i\lambda}) f_X(\lambda) \\&\equiv \Psi(e^{-i\lambda}) \overline{\Psi(e^{-i\lambda})} f_X(\lambda)\end{aligned}$$

6.3 Representation of ARMA processes

Therefore, the SDF of Y is

$$f_Y(\lambda) = |\Psi(e^{-i\lambda})|^2 f_X(\lambda) = \Psi(e^{-i\lambda})\Psi(e^{i\lambda})f_X(\lambda)$$

- $\Psi(e^{-i\lambda})$ is the **transfer function**.
- $|\Psi(e^{-i\lambda})|^2$ is the **power transfer function**.

The transfer function defines the action of the filter on the process $\{X_t\}$.

Low Pass filter

Example: Low Pass filter

A **low pass** filter removes high frequency components. To remove frequency components beyond threshold λ_C , we would require a transfer function taking the value 1 for $-\lambda_C \leq \lambda \leq \lambda_C$, that is,

$$\Psi(e^{-i\lambda}) = \begin{cases} 1 & -\lambda_C \leq \lambda \leq \lambda_C \\ 0 & \text{otherwise} \end{cases}$$

We term this filter the **ideal** low pass filter.

Low Pass filter

Example: Low Pass filter

Let

$$\psi_j = \frac{1}{2\pi} \int_{-\lambda_C}^{\lambda_C} e^{ij\lambda} d\lambda = \begin{cases} \frac{\lambda_C}{\pi} & j = 0 \\ \frac{\sin(j\lambda_C)}{j\pi} & j \neq 0 \end{cases}$$

and define

$$\Psi_q(e^{-i\lambda}) = \sum_{j=-q}^q \psi_j e^{-ij\lambda}$$

for integer q . This filter is a finite Fourier series approximation to the ideal low pass filter.

ARMA models

Example: $X_t \sim \text{ARMA}(p, q)$

Suppose $\{Z_t\} \sim \text{WN}(0, \sigma_Z^2)$, and define $\{X_t\}$ by

$$\Phi(B)X_t = \Theta(B)Z_t.$$

Thus

$$f_Z(\lambda) = \frac{\sigma_Z^2}{2\pi}$$

and as $X_t = \{\Phi(B)\}^{-1}\Theta(B)Z_t$, we have

$$f_X(\lambda) = \frac{\sigma_Z^2}{2\pi} \left| \frac{\Theta(e^{-i\lambda})}{\Phi(e^{-i\lambda})} \right|^2 = \frac{\sigma_Z^2}{2\pi} \frac{|\Theta(e^{-i\lambda})|^2}{|\Phi(e^{-i\lambda})|^2}$$

by the general result.

6.4 Model-free estimation

The SDF can be estimated from time-domain samples. Let $x_{1:n}$ be the time-domain data, and let

$$\omega_k = \frac{2\pi k}{n}$$

for k equal to

$$-\left\lfloor \frac{n-1}{2} \right\rfloor, \dots, \left\lfloor \frac{n}{2} \right\rfloor$$

The set $\{\omega_k\}$ are termed the **Fourier frequencies**.

6.4 Model-free estimation

For each k , define the $n \times 1$ vector $\mathbf{e}_k$ by

$$\mathbf{e}_k = \frac{1}{\sqrt{n}}(e^{i\omega_k}, e^{2i\omega_k}, \dots, e^{ni\omega_k})^\top$$

so that

$$\overline{\mathbf{e}}_j^\top \mathbf{e}_k = 0$$

unless $j = k$, in which case $\overline{\mathbf{e}}_k^\top \mathbf{e}_k = 1$.

6.4 Model-free estimation

$\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ form an orthonormal basis for $\mathbb{C}^n$, that is, for any n -dimensional complex number $\mathbf{x}$, we have

$$\mathbf{x} = \sum_{k=-\lfloor (n-1)/2 \rfloor}^{\lfloor n/2 \rfloor} a_k \mathbf{e}_k$$

where

$$a_k = \overline{\mathbf{e}_k}^\top \mathbf{x} = \frac{1}{\sqrt{n}} \sum_{t=1}^n x_t e^{-it\omega_k}$$

The vector $(a_1, \dots, a_n)^\top$ is the **discrete Fourier transform** of $\mathbf{x}$.

6.4 Model-free estimation

Note that

$$x_t = \sum_{k=-\lfloor (n-1)/2 \rfloor}^{\lfloor n/2 \rfloor} a_k [\cos(\omega_k t) + i \sin(\omega_k t)]$$

is the discrete version of the spectral representation theorem.

The Periodogram

The **periodogram**, $\hat{I}_n(\lambda)$ is defined by

$$\hat{I}_n(\lambda) = \frac{1}{n} \left| \sum_{t=1}^n \mathbf{x}_t e^{-it\lambda} \right|^2 \quad -\pi < \lambda \leq \pi.$$

Note that $\hat{I}_n(\omega_k) = |a_k|^2$, and

$$\bar{\mathbf{x}}^\top \mathbf{x} = \sum_{k=-\lfloor (n-1)/2 \rfloor}^{\lfloor n/2 \rfloor} |a_k|^2 = \sum_{k=-\lfloor (n-1)/2 \rfloor}^{\lfloor n/2 \rfloor} \hat{I}_n(\omega_k)$$

which gives a decomposition of the total variation in $\mathbf{x}$ via components at the ω_k .

The Periodogram

Note that

$$\begin{aligned}\hat{I}_n(\omega_k) &= \left(\frac{1}{\sqrt{n}} \sum_{t=1}^n x_t e^{-it\omega_k} \right) \left(\frac{1}{\sqrt{n}} \sum_{s=1}^n x_s e^{is\omega_k} \right) \\&= \frac{1}{n} \sum_{t=1}^n \sum_{s=1}^n x_t x_s e^{-i(t-s)\omega_k} \\&= \frac{1}{n} \sum_{t=1}^n \sum_{s=1}^n (x_t - \bar{x})(x_s - \bar{x}) e^{-i(t-s)\omega_k} \\&= \sum_{h=-(n-1)}^{n-1} \hat{\gamma}_n(h) e^{-ih\omega_k} \quad (h = s - t)\end{aligned}$$

$$\text{as } \sum_{t=1}^n e^{-it\omega_k} = 0 \text{ if } \omega_k \neq 0.$$

The Periodogram

Thus $\hat{I}_n(\lambda)$ is a sample-based estimate of $2\pi f_X(\lambda)$; it is **not** consistent, however. For $0 < \lambda_1 < \dots < \lambda_m < \pi$,

$$(\hat{I}_n(\lambda_1), \dots, \hat{I}_n(\lambda_m))^{\top} \xrightarrow{d} (V_1, \dots, V_n)^{\top}$$

as $n \rightarrow \infty$, where

$$V_j \sim \text{Exponential}(1/(2\pi f_X(\lambda_j))).$$

That is, as $n \rightarrow \infty$

$$\mathbb{E}[\hat{I}_n(\lambda_j)] \rightarrow 2\pi f_X(\lambda_j)$$

so $\hat{I}_n(\lambda_j)$ is asymptotically unbiased, but $\hat{I}_n(\lambda_j)$ converges in distribution, not in probability or almost surely.

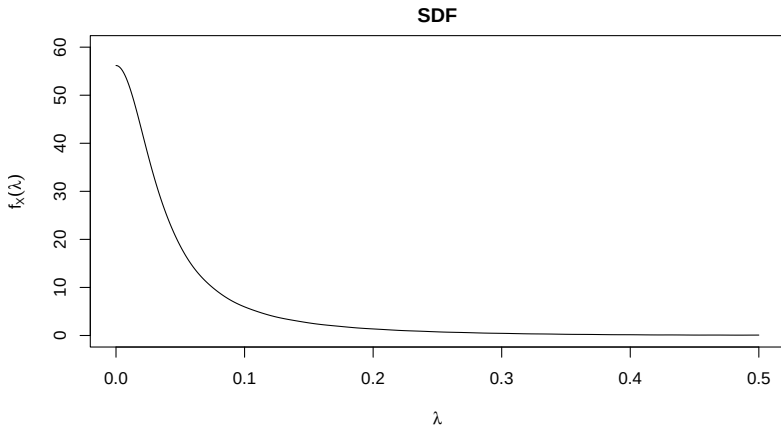
The Periodogram

Example: ARMA(1,1)

```
acvf.vals<-tacvfARMA(phi = 0.8, theta = -0.5, maxLag = 30)
lamvec<-seq(0,0.5,by=0.001)
sdf.vec<-lamvec*0
hvec<-0:30
for(j in 1:length(lamvec)){
  sdf.vec[j]<-sum(exp(-1i*2*pi*lamvec[j]*hvec[-1]))*acvf.vals[2:31])+acvf.vals[1]+
  sum(exp(1i*2*pi*lamvec[j]*hvec[-1]))*acvf.vals[2:31])
}
```

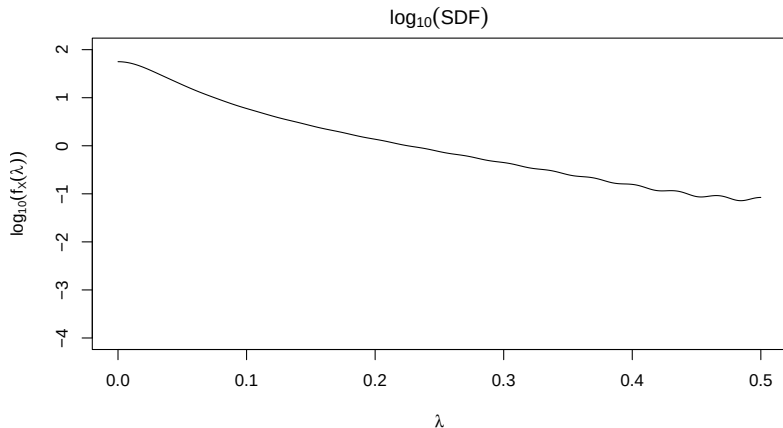
The Periodogram

```
par(mar=c(4,4,2,2));plot(lamvec,sdf.vec,type='l',xlab=expression(lambda),  
ylab=expression(f[X](lambda)), ylim=range(0,60),main = 'SDF')
```



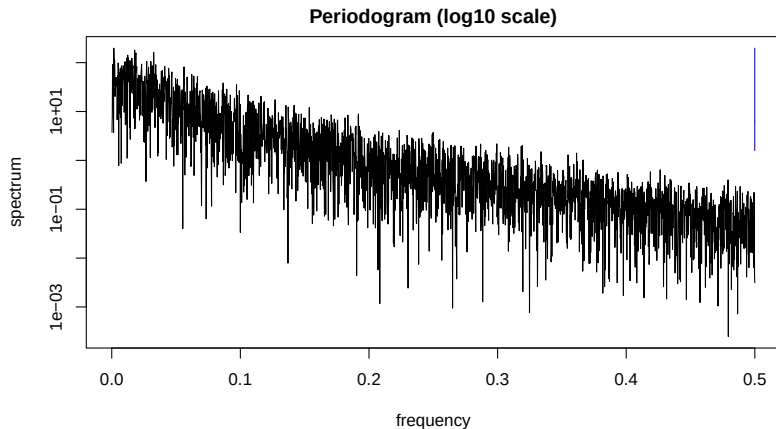
The Periodogram

```
par(mar=c(4,4,2,2));plot(lamvec,log10(sdf.vec),type='l',xlab=expression(lambda),  
ylab=expression(log[10](f[X](lambda))), ylim=range(-4,2),main = expression(log[10](SDF)))
```



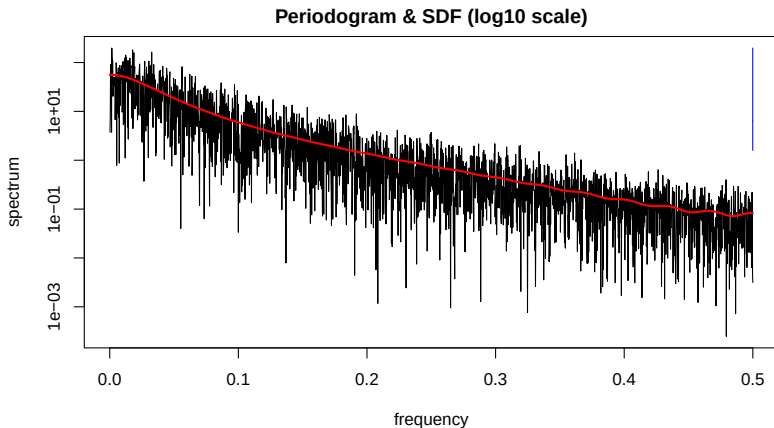
The Periodogram

```
set.seed(3); Xt<-arima.sim(n=5000,model=list(ar=0.8,ma=0.5))  
par(mar=c(4,4,2,2));spectrum(Xt, main='Periodogram (log10 scale)', taper=0,detrend=FALSE)
```

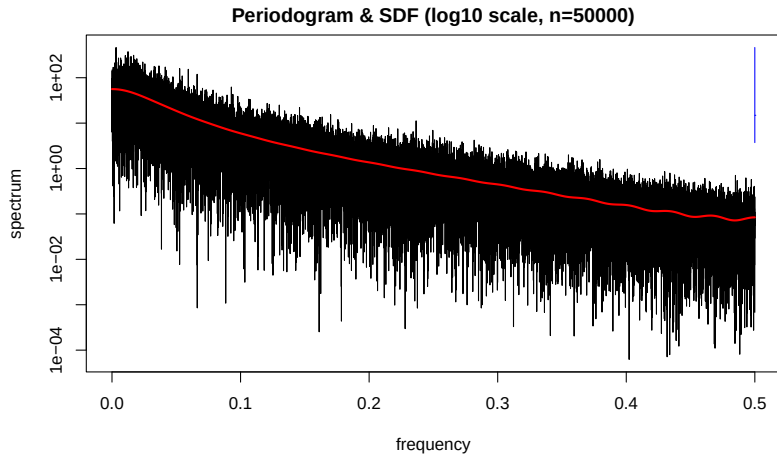


The Periodogram

```
par(mar=c(4,4,2,2));spectrum(Xt, main='Periodogram & SDF (log10 scale)', taper=0,detrend=FALSE)
lines(lamvec,sdf.vec,col='red',lwd=2)
```



The Periodogram

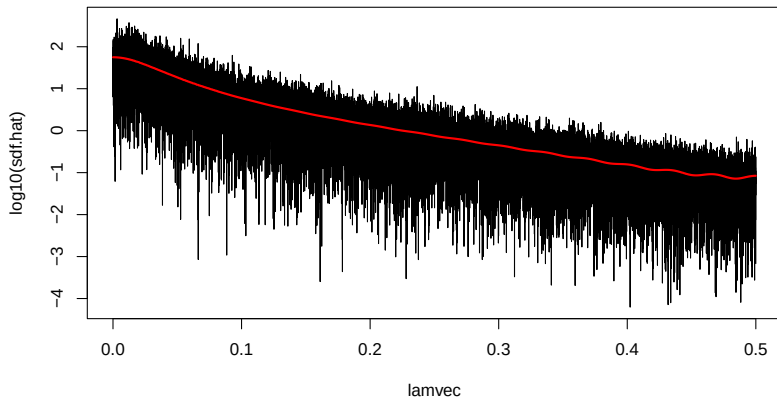


The Periodogram

```
sdf.hat<-spectrum(Xt, taper=0,detrend=FALSE,plot=FALSE)$spec
n0<-length(sdf.hat)
lamvec<-c(1:n0)/(2*n0)
sdf.vec<-lamvec*0
hvec<-0:30
for(j in 1:length(lamvec)){
  sdf.vec[j]<-sum(exp(-1i*2*pi*lamvec[j]*hvec[-1])*acvf.vals[2:31])+acvf.vals[1]+
    sum(exp(1i*2*pi*lamvec[j]*hvec[-1])*acvf.vals[2:31])
}
```

The Periodogram

```
par(mar=c(4,4,2,2));plot(lamvec,log10(sdf.hat),type='l')  
lines(lamvec,log10(sdf.vec),col='red',lwd=2)
```



The Periodogram

If in the above notation,

$$V_j \sim \textit{Exponential}(1/(2\pi f_X(\lambda_j))).$$

then

$$\frac{V_j}{2\pi f_X(\lambda_j)} \sim \textit{Exponential}(1)$$

which we can verify in the sample.

Note that in the previous calculations, the factor of 2π is already incorporated to account for the output of spectrum.

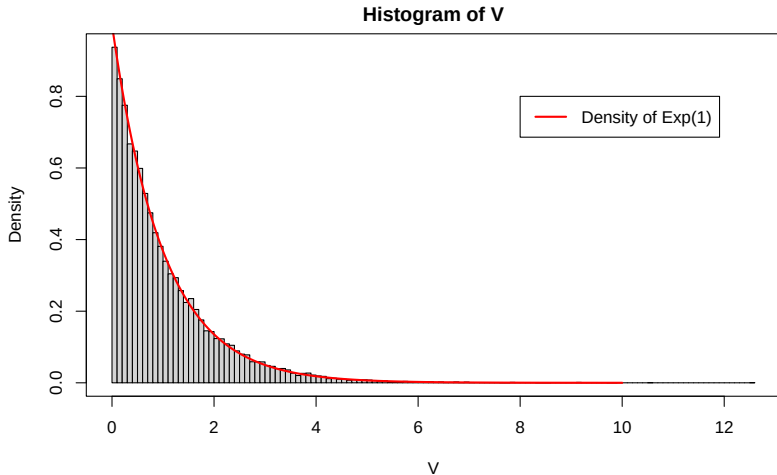
The Periodogram

```
Vvec<-as.numeric(sdf.hat/sdf.vec)
mean(Vvec); var(Vvec)

## [1] 1.0135
## [1] 1.025988

par(mar=c(4,4,2,2))
hist(Vvec,nclass=100, main='Histogram of V', xlab='V', freq=FALSE)
xv<-seq(0,10,by=0.001)
lines(xv,dexp(xv),col='red',lwd=2)
legend(8,0.8,c('Density of Exp(1)'),lwd=2,col='red')
box()
```

The Periodogram



The Periodogram

To obtain a consistent estimator, a local averaging process is used: define

$$\hat{f}_n(\lambda) = \frac{1}{2\pi} \sum_{|j| \leq m_n} w_n(j) \hat{I}_n(g(n, \lambda) + 2\pi j/n)$$

where **bandwidths** m_n satisfy $m_n \rightarrow \infty$, $m_n/n \rightarrow 0$ as $n \rightarrow \infty$, and weights $w_n(j)$ satisfy $w_n(j) = w_n(-j) \geq 0$,

$$\sum_{|j| \leq m_n} w_n(j) = 1$$

and $w_n^2(j) \rightarrow 0$ as $n \rightarrow \infty$.

Example: $m_n = \sqrt{n}$, $w_n(j) = 1/(2m_n + 1)$ for $|j| \leq m_n$.

6.5 Likelihood-based estimation

For parametric models such as ARMA models with Gaussian errors, likelihood-based inference can be carried out using a large sample approximation to the time domain likelihood known as the **Whittle likelihood**.

Recall that for the periodogram, for any λ , as $n \rightarrow \infty$

$$\hat{I}_n(\lambda) \xrightarrow{d} V \sim \text{Exponential}(1/(2\pi f_X(\lambda)))$$

If $\lambda_1, \dots, \lambda_n$ are chosen to be the Fourier frequencies, then

$$V_1, \dots, V_n$$

are approximately independent.

6.5 Likelihood-based estimation

The Whittle likelihood is then

$$\mathcal{L}(\theta) = \prod_j \frac{1}{2\pi f_x(\omega_j; \theta)} \exp \left\{ -\frac{\hat{I}_n(\omega_j)}{2\pi f_x(\omega_j; \theta)} \right\}$$

where $\omega_j = 2\pi j/n$, and

$$j \in \left\{ -\left\lfloor \frac{n-1}{2} \right\rfloor, \dots, \left\lfloor \frac{n}{2} \right\rfloor \right\}$$

This (asymptotic) likelihood can then be maximized to estimate θ .

6.6 Long memory models

For a stationary ARMA model, recall that the decay in ACF is asymptotically **exponentially fast**: for h large enough,

$$\rho_X(h) \approx C|\xi_1|^h$$

for some constant C , say, where ξ_1 is reciprocal root of the AR polynomial with modulus closest to 1.

6.6 Long memory models

Let $\{X_t\}$ be a stationary process such that for some $\alpha \in (0, 1)$ and constant $c > 0$ such that

$$\lim_{h \rightarrow \infty} h^\alpha \rho_X(h) = c$$

That is, the decay rate is **polynomial** in h , and therefore relatively slow.

$\{X_t\}$ is a **long memory** or **long range dependent** process.

Fractional Integration

Recall the ARIMA(p, d, q) model, for $d = 1, 2, \dots$,

$$(1 - B)^d \Phi(B) X_t = \Theta(B) Z_t$$

Now suppose that $-0.5 < d < 0.5$ is a **non-integer** value.

Fractional Integration

We have the binomial expansion for $-0.5 < d < 0.5$

$$(1 - B)^d = \sum_{j=0}^{\infty} \psi_j B^j$$

where, for $j = 1, 2, \dots$

$$\psi_j = \prod_{k=1}^j \left(\frac{k-1-d}{k} \right) = \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)}$$

where $\Gamma(\cdot)$ is the Gamma function.

Fractional Integration

Let $\{Z_t\} \sim \text{WN}(0, \sigma^2)$, and $W_t = (1 - B)^{-d} Z_t$, so that

$$\Phi(B)X_t = \Theta(B)W_t$$

gives an ARMA(p, q) model driven by $\{W_t\}$.

$\{W_t\}$ is **fractionally integrated white noise**. It is a stationary process with

$$\gamma_W(0) = \frac{\sigma_Z^2 \Gamma(1 - 2d)}{\{\Gamma(1 - d)\}^2}$$

and

$$\rho_W(h) = \frac{\Gamma(h + d)\Gamma(1 - d)}{\Gamma(h - d + 1)\Gamma(d)} = \prod_{k=1}^h \left(\frac{k - 1 + d}{k - d} \right)$$

Fractional Integration

The ACVF of $\{X_t\}$ can be computed using the usual approach

$$\gamma_X(h) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \psi_j \psi_k \gamma_W(h + j - k)$$

where $\{\psi_j\}$ are the linear process coefficients corresponding to the ARMA process with polynomials $\Phi(B)$ and $\Theta(B)$.

Spectral density for Fractional ARIMA

If $\{X_t\} \sim \text{ARIMA}(p, d, q)$ with $-1/2 < d < 1/2$, then

$$f_X(\lambda) = \frac{\sigma_Z^2}{2\pi} \frac{|\Theta(e^{-i\lambda})|^2}{|\Phi(e^{-i\lambda})|^2} \frac{1}{|1 - e^{-i\lambda}|^{2d}}$$

If $p = q = 0$,

$$f_X(\lambda) = \frac{\sigma_Z^2}{2\pi} \frac{1}{|1 - e^{-i\lambda}|^{2d}}$$

and as $\lambda \rightarrow 0$

$$f_X(\lambda) \approx \frac{\sigma_Z^2}{2\pi|\lambda|^{2d}}$$

that is, the SDF is unbounded at zero.

Fractional White Noise

If $\{W_t\} \sim \text{ARIMA}(0, d, 0)$, that is,

$$W_t = (1 - B)^{-d} Z_t$$

then

- MA(∞) representation

$$W_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \quad \psi_j = \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)}$$

- AR(∞) representation

$$Z_t = \sum_{j=0}^{\infty} \pi_j W_{t-j} \quad \pi_j = \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)}$$

Fractional White Noise

For large j

- MA(∞) representation

$$\psi_j \approx \frac{j^{d-1}}{\Gamma(d)}$$

- AR(∞) representation

$$\pi_j \approx \frac{j^{-d-1}}{\Gamma(-d)}$$

Fractional White Noise

If $\{X_t\} \sim \text{ARIMA}(p, d, q)$, write

$$(1 - B)^d \Phi(B) X_t = \Theta(B) Z_t$$

and suppose $0 < d < 0.5$. Then for large h

$$\gamma_X(h) \approx C|h|^{2d-1}.$$