

MATH 545: Introduction to Time Series Analysis

Practice Final: Solutions

Winter 2025

Note: I have included in some of the answers some of the key theoretical calculations that are not asked for in the questions to give a reminder of how they work in context. Also, in some cases, alternative equivalent answers are possible.

MATH 545: Specific Instructions.

Please write your answers on the exam in the boxes provided for each question; only answers written inside the correct boxes will be marked. Extra space for answers can be found at the end of the exam paper. Please write clearly.

This paper contains five questions. Each question carries 20 marks. Credit will be given for all questions attempted. The total mark available is 100 but rescaling of the final mark may occur.

Throughout the exam, $\{Z_t\}$ refers to a white noise process with variance σ_Z^2 unless otherwise stated; when stationarity is referred to, it is weak stationarity.

Only non-programmable and non text-storing calculators may be used. Dictionaries and translation dictionaries are permitted. A formula sheet is provided.

1. (a) What is meant by saying that a stochastic process is stationary?

2 MARKS

Q1a

As stated in the rubric, we focus only on weak stationarity: for stochastic process $\{X_t\}$ to be stationary, we require that the quantities

$$\mathbb{E}[X_t] \quad \text{Cov}[X_t, X_{t+h}], h = 0, \pm 1, \pm 2, \dots$$

are finite and do not depend on t .

(b) For an $\text{AR}(p)$ process defined by the equation

$$X_t = \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} + Z_t$$

what is the condition on the coefficients $\phi_1, \dots, \phi_p$ that is required for the process to be stationary?

4 MARKS

Q1b

We can write the $\text{AR}(p)$ process as

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \dots - \phi_p X_{t-p} = (1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p) X_t = Z_t.$$

The condition for stationarity is that the roots of the characteristic polynomial

$$1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p = \Phi(z)$$

do not lie on the unit circle, that is, have modulus not equal to one. For a causal process, we need that the reciprocal roots $\xi_1, \dots, \xi_p$ where

$$\Phi(z) = \prod_{j=1}^p (1 - \xi_j z)$$

lie **inside** the unit circle, or equivalently, that the roots lie outside the unit circle.

(c) Consider the following AR(2) model,

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t.$$

that is known to be stationary.

(i) Using the backward shift operator B , show that this model can be written in the infinite MA or linear process form

$$X_t = \sum_{j=0}^{\infty} \frac{\xi_1^{j+1} - \xi_2^{j+1}}{\xi_1 - \xi_2} Z_{t-j},$$

where

$$(1 - \phi_1 B - \phi_2 B^2) = (1 - \xi_1 B)(1 - \xi_2 B).$$

8 MARKS

Q1ci

If

$$(1 - \phi_1 B - \phi_2 B^2)X_t = (1 - \xi_1 B)(1 - \xi_2 B)X_t = Z_t$$

then

$$X_t = \{(1 - \xi_1 B)(1 - \xi_2 B)\}^{-1} Z_t = \left(\sum_{j=0}^{\infty} \xi_1^j B^j \right) \left(\sum_{k=0}^{\infty} \xi_2^k B^k \right) Z_t = \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \xi_1^j \xi_2^k B^{j+k} \right) Z_t$$

Thus

$$X_t = \left(\sum_{j=0}^{\infty} \psi_j B^j \right) Z_t$$

where

$$\psi_j = \sum_{k=0}^j \xi_1^k \xi_2^{j-k} \equiv \frac{\xi_2^{j+1} - \xi_1^{j+1}}{\xi_2 - \xi_1}.$$

Q1ci

Equivalently for arbitrary complex z ,

$$\frac{1}{(1 - \xi_1 z)(1 - \xi_2 z)} = \frac{A}{1 - \xi_1 z} + \frac{C}{1 - \xi_2 z}$$

from which we obtain,

$$\begin{aligned} A + C = 1 &\Rightarrow A\xi_2 = (1 - C)\xi_2 \\ -(A\xi_2 + C\xi_1) = 0 &\Rightarrow (1 - C)\xi_2 = -C\xi_1 \\ &\Rightarrow C = \xi_2/(\xi_2 - \xi_1). \end{aligned}$$

Hence, $A = 1 - C = -\xi_1/(\xi_2 - \xi_1)$. Thus

$$\frac{1}{(1 - \xi_1 z)(1 - \xi_2 z)} = \left(\frac{1}{\xi_2 - \xi_1} \right) \left[\frac{\xi_2}{1 - \xi_2 z} - \frac{\xi_1}{1 - \xi_1 z} \right],$$

and hence,

$$\begin{aligned} \{(1 - \xi_1 B)(1 - \xi_2 B)\}^{-1} &= \left(\frac{1}{\xi_2 - \xi_1} \right) [\xi_2(1 - \xi_2 B)^{-1} - \xi_1(1 - \xi_1 B)^{-1}] \\ &= \left(\frac{1}{\xi_2 - \xi_1} \right) \left[\xi_2 \sum_{j=0}^{\infty} \xi_2^j B^j - \xi_1 \sum_{j=0}^{\infty} \xi_1^j B^j \right] \\ &= \left(\frac{1}{\xi_2 - \xi_1} \right) \sum_{j=0}^{\infty} (\xi_2^{j+1} - \xi_1^{j+1}) B^j \end{aligned}$$

so that

$$X_t = \sum_{j=0}^{\infty} \frac{\xi_2^{j+1} - \xi_1^{j+1}}{\xi_2 - \xi_1} Z_{t-j}.$$

- (ii) Use the expression derived in part (c) (i) to verify that the condition given in part (b) does indeed imply stationarity for $\{X_t\}$. 6 MARKS

Q1cii

Using the linear process representation, and properties of the zero-mean white noise process, we have

$$\mathbb{E}[X_t] = 0, \quad \text{Var}[X_t] = \frac{1}{(\xi_2 - \xi_1)^2} \sum_{k=0}^{\infty} (\xi_2^{k+1} - \xi_1^{k+1})^2 \sigma_Z^2,$$

and

$$\begin{aligned} \gamma_X(h) &= \mathbb{E}[X_t X_{t+h}] \\ &= \frac{1}{(\xi_2 - \xi_1)^2} \mathbb{E} \left[\left\{ \sum_{k=0}^{\infty} (\xi_2^{k+1} - \xi_1^{k+1}) Z_{t-k} \right\} \left\{ \sum_{j=0}^{\infty} (\xi_2^{j+1} - \xi_1^{j+1}) Z_{t+h-j} \right\} \right] \\ &= \frac{1}{(\xi_2 - \xi_1)^2} \sum_{k=0}^{\infty} (\xi_2^{k+1} - \xi_1^{k+1}) (\xi_2^{h+k+1} - \xi_1^{h+k+1}) \sigma_Z^2, \end{aligned}$$

as $\mathbb{E}[Z_{t-k} Z_{t+h-j}]$ is only non-zero when $j = h + k$. For stationarity for this causal process, the above variance and covariance terms must be finite, this occurs when the modulus of the multipliers in the geometric progressions are less than one, this is equivalent to

$$|\xi_1| < 1 \quad \text{and} \quad |\xi_2| < 1.$$

The condition in part (b) is that the roots of $\Phi(z)$ lie outside the unit circle, *i.e.*

$$\Phi(z) = (1 - \phi_1 z - \phi_2 z^2) = (1 - \xi_1 z)(1 - \xi_2 z) = 0$$

$$\Rightarrow z = \frac{1}{\xi_1}, \quad z = \frac{1}{\xi_2}$$

$$|z| > 1 \Rightarrow |\xi_1| < 1 \quad \text{and} \quad |\xi_2| < 1,$$

as required.

2. The process $\{X_t\}$ is given by,

$$X_t = \alpha + \beta t + Y_t,$$

where $\{Y_t\}$ is a stationary process.

(a) Show that differencing $\{X_t\}$ twice can be used to produce a new process $\{X_t^*\}$

$$X_t^* = \nabla^2 X_t = (1 - B)^2 X_t,$$

which does not depend on α or β . Determine whether $\{X_t^*\}$ is stationary. 6 MARKS

Q2a

$$\begin{aligned} X_t^* &= \nabla^2 X_t = (1 - B)^2 X_t = (1 - 2B + B^2) X_t = X_t - 2X_{t-1} + X_{t-2} \\ &= (\alpha + \beta t + Y_t) - 2(\alpha + \beta(t-1) + Y_{t-1}) + (\alpha + \beta(t-2) + Y_{t-2}) \\ &= Y_t - 2Y_{t-1} + Y_{t-2} = \nabla^2 Y_t. \end{aligned}$$

which does not depend on α or β , as required.

If we can show that taking first differences of a stationary process produces a stationary process, then we will have shown that $\{X_t^*\}$ is also stationary (as $X_t^* = \nabla^2 Y_t = \nabla(\nabla Y_t)$ and Y_t is stationary). $\{Y_t\}$ is stationary, therefore

$$\mathbb{E}[Y_t] = \mu, \quad \text{Var}[Y_t] = \sigma_Y^2, \quad \text{Cov}[Y_t, Y_{t+h}] = \gamma_Y(h).$$

are finite constants, not depending on t . Let $X_t^\dagger = \nabla Y_t$,

$$\begin{aligned} \mathbb{E}[X_t^\dagger] &= \mathbb{E}[Y_t - Y_{t-1}] = 0; \\ \text{Var}[X_t^\dagger] &= \mathbb{E}[(Y_t - Y_{t-1})^2] = \sigma_Y^2 - 2\gamma_Y(1) + \sigma_Y^2 = 2(\sigma_Y^2 - \gamma_Y(1)); \\ \text{Cov}[X_t^\dagger, X_{t+h}^\dagger] &= \mathbb{E}[(Y_t - Y_{t-1})(Y_{t+h} - Y_{t+h-1})] \\ &= \gamma_Y(h) - \gamma_Y(h-1) - \gamma_Y(h+1) + \gamma_Y(h) \\ &= 2\gamma_Y(h) - \gamma_Y(h-1) - \gamma_Y(h+1). \end{aligned}$$

so, the mean and variance of the first difference of a stationary process are also finite constants and the covariances are finite quantities which do not depend on t

$\Rightarrow X_t^\dagger$ is stationary

$\Rightarrow \{X_t^*\}$ where $X_t^* = \nabla X_t^\dagger$ is stationary.

- (b) (i) What property/properties of a linear filter, $\Psi(B)$ with coefficients $\{\psi_j\}$ are required for the process $\{Y_t\}$,

$$Y_t = \Psi(B)Z_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$$

to be well-defined in *mean-square* terms ? Give the definition of mean-square equality of random variables. 4 MARKS

Q2bi

We need that

$$\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$$

which ensures convergence in mean-square of partial sums

$$\lim_{k \rightarrow \infty} \sum_{j=-k}^k \psi_j Z_{t-j} = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} = \Psi(B)Z_t$$

say. Mean-square equality is defined by saying that

$$X_t \stackrel{\text{m.s.}}{=} Y_t \iff \mathbb{E}[(X_t - Y_t)^2] = 0.$$

- (ii) If $\{U_t\}$ defined by $U_t = \Psi(B)Z_t$, derive the general form of the autocorrelation function (ACF), $\rho_U(h)$, in terms of the coefficients of $\Psi(B)$. 4 MARKS

Q2bii

$\{U_t\}$ is zero mean, and

$$\mathbb{E}[U_t U_{t+h}] = \mathbb{E} \left[\left(\sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} \right) \left(\sum_{k=-\infty}^{\infty} \psi_k Z_{t+h-k} \right) \right] = \mathbb{E} \left[\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k Z_{t-j} Z_{t+h-k} \right]$$

in which only the terms where $j = k - h$, or $k = j + h$ contributing. In this case $\mathbb{E}[Z_{t-j} Z_{t+h-k}] = \sigma_Z^2$. Therefore

$$\gamma_U(h) = \sigma_Z^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \quad \rho_U(h) = \frac{\gamma_U(h)}{\gamma_U(0)} = \frac{\sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h}}{\sum_{j=-\infty}^{\infty} \psi_j^2} \quad h = 0, \pm 1, \pm 2, \dots$$

(iii) If $\{U_t\}$ is an MA(2) process given by,

$$U_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}.$$

Determine the ACVF for $\{U_t\}$ in terms of θ_1, θ_2 and σ_Z^2 .

6 MARKS

Q2c

Direct computation technique: we have

$$U_t U_{t+h} = (Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2})(Z_{t+h} + \theta_1 Z_{t+h-1} + \theta_2 Z_{t+h-2})$$

so therefore

$$\mathbb{E}[U_t U_{t+h}] = \mathbb{E}[(Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2})(Z_{t+h} + \theta_1 Z_{t+h-1} + \theta_2 Z_{t+h-2})]$$

For $h = 0$,

$$\gamma_U(0) = \mathbb{E}[U_t^2] = \sigma_Z^2(1 + \theta_1^2 + \theta_2^2).$$

For $h = 1$,

$$\gamma_U(1) = \mathbb{E}[U_t U_{t+1}] = \sigma_Z^2(\theta_1 + \theta_1 \theta_2).$$

For $h = 2$,

$$\gamma_U(2) = \mathbb{E}[U_t U_{t+2}] = \sigma_Z^2 \theta_2.$$

and $\gamma_U(h) = 0, h > 2$.

3. (a) A stationary stochastic process is termed *invertible* if it admits a *causal* $AR(\infty)$ representation.

(i) Define the term causal used in this definition.

2 MARKS

Q3ai

An AR process $\{X_t\}$ is causal if it can be written as a linear combination of random quantities Z_s or equivalently X_s , with indices $s \leq t$ only. That is, we must have

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \quad \text{or} \quad X_t = \sum_{j=1}^{\infty} \phi_j X_{t-j} + Z_t$$

for some coefficients $\{\psi_j\}$ or $\{\phi_j\}$.

(ii) Show that any stationary $AR(p)$ process is invertible.

3 MARKS

Q3aii

The stationary $AR(p)$ process

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t$$

necessarily satisfies the above condition, as it can be written as a finite version of the expansion in (a)

(iii) Write down a condition, involving the roots or reciprocal roots of an equation, for an $MA(q)$ process to be invertible.

3 MARKS

Q3aiii

We can write the $MA(q)$ process as,

$$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q} = (1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q) Z_t.$$

The condition for invertibility is that the roots of the MA polynomial

$$1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q = \Theta(z)$$

lie **outside** the unit circle, or equivalently that the reciprocal roots lie **inside** the unit circle.

- (b) (i) Determine the autocovariance function (ACVF), $\gamma_X(h)$, $h = 0, \pm 1, \pm 2, \dots$ for the MA(1) process,

$$X_t = Z_t + \theta_1 Z_{t-1}.$$

4 MARKS

Q3bi

By direct calculation

$$\gamma_X(h) = \mathbb{E}[X_t X_{t+h}] = \mathbb{E}[(Z_t + \theta_1 Z_{t-1})(Z_{t+h} + \theta_1 Z_{t+h-1})] = \begin{cases} \sigma_Z^2(1 + \theta_1^2) & h = 0 \\ \sigma_Z^2 \theta_1 & h = 1 \\ 0 & h \geq 2 \end{cases}$$

- (ii) When modelling a process $\{X_t\}$ as an MA(1) process, show that there is more than one such process with $\gamma_X(0) = 5.0$ and $\gamma_X(1) = 2.0$. Verify that only one of these processes is invertible. 8 MARKS

Q3bii

We need to consider the solution of the simultaneous equations

$$\sigma_Z^2(1 + \theta_1^2) = 5 \quad \sigma_Z^2\theta_1 = 2$$

that is

$$2(1 + \theta_1^2) = 5\theta_1$$

or

$$\theta_1^2 - \frac{5}{2}\theta_1 + 1 = 0.$$

The solutions to this quadratic is

$$\theta_1 = \frac{5}{4} \pm \frac{1}{2}\sqrt{\frac{25}{4} - 4} = \frac{5}{4} \pm \frac{3}{4}$$

and precisely one solution is less than one in modulus. This root, $\theta_1 = 1/2$, defines an invertible process, the other $\theta_1 = 2$, is not invertible.

4. (a) Explain the difference between *deterministic* and *stochastic* seasonal behaviour, and describe a model or method designed to handle each type of behaviour. 6 MARKS

Q4a

Essentially *deterministic* seasonality indicates a periodic variation in mean - this can be handled using indicator variables or using a periodic parametric function. For example, for period d , we use $s_1, \dots, s_d$ such that

$$s_{t+d} = s_t \quad \forall t \quad \text{and} \quad \sum_{t=1}^d s_t = 0$$

which may be estimated from the data using least squares and then subtracted from the original data. We may also use harmonic regression, for example

$$s_t = \beta_1 \cos(\omega t) + \beta_2 \sin(\omega t).$$

Stochastic seasonality refers to a component which can be written as a filtered version of a white noise process e.g. for period s ,

$$(1 - \theta B^s)(1 - B^s)^D Z_t$$

where $D = 0, 1, \dots$ indicates a degree of differencing necessary to induce stationarity, and $\theta \neq 1$. In the stochastic version, there is **not** necessarily systematic variation in mean.

(b) Write down the AR and/or MA polynomials for the following SARIMA models, which are written in the usual notation as

(i) SARIMA(0, 1, 1) $\times$ (1, 1, 0)₁₂. 3 MARKS

(ii) SARIMA(1, 0, 0) $\times$ (1, 0, 1)₄. 3 MARKS

(iii) SARIMA(0, 1, 0) $\times$ (1, 0, 0)₇. 3 MARKS

Q4b

(i) SARIMA(0, 1, 1) $\times$ (1, 1, 0)₁₂: Model is

$$(1 - \phi_{S1}B^{12})(1 - B^{12})(1 - B)X_t = (1 + \theta_1B)Z_t$$

so

$$\phi(z) = 1 - z - (1 + \phi_{S1})z^{12} + (1 + \phi_{S1})z^{13} + \phi_{S1}z^{24} - \phi_{S1}z^{25} \quad \theta(z) = 1 + \theta_1z$$

(ii) SARIMA(1, 0, 0)(1, 0, 1)₄: Model is

$$(1 - \phi_{S1}B^4)(1 - \phi_1B)X_t = (1 + \theta_{S1}B^4)Z_t$$

so

$$\phi(z) = 1 - \phi_1z - \phi_{S1}z^4 + \phi_1\phi_{S1}z^5 \quad \theta(z) = 1 + \theta_{S1}z^4$$

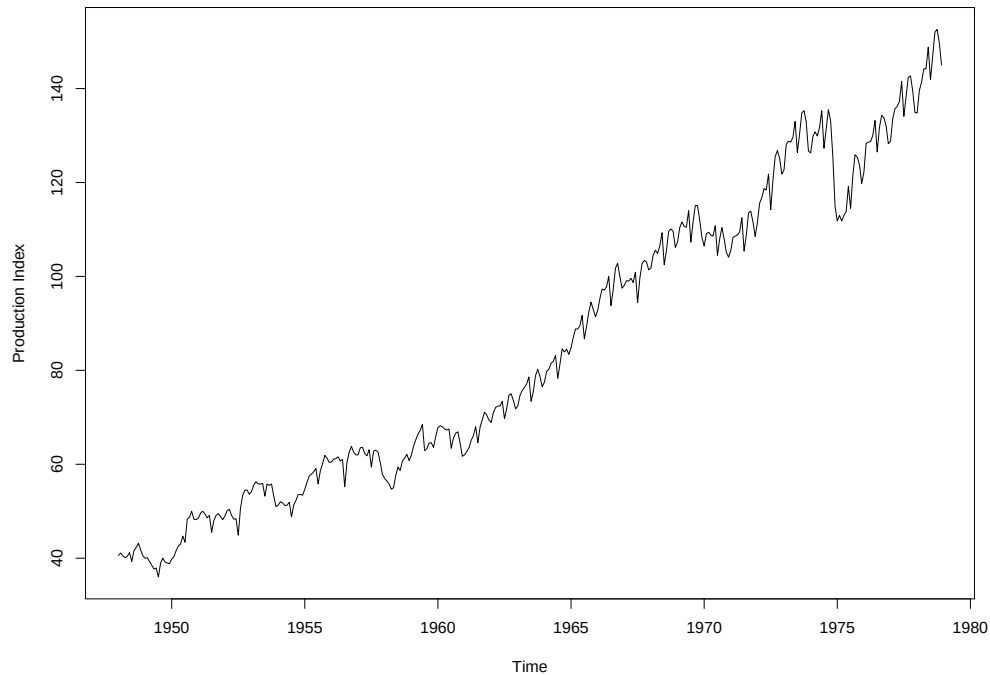
(iii) SARIMA(0, 1, 0)(1, 0, 0)₇: Model is

$$(1 - \phi_{S1}B^7)(1 - B)X_t = Z_t$$

so

$$\phi(z) = 1 - z - \phi_{S1}z^7 + \phi_{S1}z^8 \quad \theta(z) = 1$$

- (c) The plot below depicts monthly US Federal Reserve Board production index values, 1948-1978 ($n = 372$ months).



- (i) Is this a realization of a stationary process ? Justify your answer. 2 MARKS

Q4ci

This is clearly not stationary, as the mean is not constant. There may also be periodic behaviour.

- (ii) Give one suggestion for a data transformation that may make the resulting transformed series stationary ? 2 MARKS

Q4cii

The most obvious elementary transformation is to take first differences.

- (iii) Outline an analysis strategy that you would use to analyze these data if you were attempting to construct a realistic model for forecasting. 4 MARKS

Q4ciii

Could perform a classic decomposition

$$Y_t = m_t + s_t + X_t$$

where m_t is a time-varying mean, and s_t is a seasonal term that may be estimated using least squares. $\{X_t\}$ is a stationary stochastic process. After estimating m_t and s_t , which may be modelled parametrically or semi-parametrically and estimated using least squares, the residual series

$$\hat{X}_t = Y_t - \hat{m}_t - \hat{s}_t$$

can be analyzed by inspecting plots for homogeneity in variance (plotting the time series of residuals), and acf plots. An alternative strategy would be to use the state space model formulation with local linear structure and seasonality, that is setting

$$\begin{aligned} Y_t &= M_t + S_t + V_t \\ M_t &= M_{t-1} + B_{t-1} + W_{1,(t-1)} \\ B_t &= B_{t-1} + W_{2,(t-1)} \end{aligned}$$

where $\{M_t\}$ defines the local mean, $\{B_t\}$ defines the local slope, with $\{S_t\}$ defined by the recursion, for some d

$$S_{t+1} = (-S_t - S_{t-1} - \cdots - S_{t-d+2}) + W_t$$

and $\{V_t\}$, $\{W_t\}$, $\{W_{1,t}\}$ and $\{W_{2,t}\}$ are independent white noise processes. This model can be estimated using the Kalman filter. After fitting, we can inspect the residual series

$$\widehat{W}_t = Y_t - \widehat{M}_t - \widehat{S}_t$$

as above, where the fitted values are computed as the filtering, prediction, or smoothing estimated states from the Kalman filter.

5. Consider the following state space model

$$Y_t = X_t + V_t$$

$$X_{t+1} = \beta X_t + W_t$$

where $\{V_t\}$ and $\{W_t\}$ are independent, zero-mean Gaussian IID processes with variances σ_V^2 and σ_W^2 respectively, and β is a real-valued parameter.

(a) Derive the recursion for the filtering and prediction densities

$$p(x_t|y_{1:t}) \quad p(x_{t+1}|y_{1:t})$$

in this model, that is, show that these distributions are Normal, with means and variances determined by the means and variances of the preceding densities

$$p(x_{t-1}|y_{1:(t-1)}) \quad p(x_t|y_{1:(t-1)})$$

which are also Normal.

8 MARKS

Hint: the univariate Gaussian density with mean m and variance v takes the form

$$f(x|m, v) = \left(\frac{1}{2\pi v}\right)^{1/2} \exp\left\{-\frac{1}{2v}(x - m)^2\right\}.$$

so that, for example, in the model in (c),

$$f(x_{t+1}|x_t) = \left(\frac{1}{2\pi\sigma_W^2}\right)^{1/2} \exp\left\{-\frac{1}{2\sigma_W^2}(x_{t+1} - \beta x_t)^2\right\}.$$

and

$$f(y_t|x_t) = \left(\frac{1}{2\pi\sigma_V^2}\right)^{1/2} \exp\left\{-\frac{1}{2\sigma_V^2}(y_t - x_t)^2\right\}.$$

The following completing-the-square identity may be useful:

$$A(x - a)^2 + B(x - b)^2 = (A + B)(x - m)^2 + \frac{AB}{A + B}(a - b)^2$$

where

$$m = \frac{Aa + Bb}{A + B}.$$

Q5a

For the prediction step, we have that

$$\begin{aligned}
 p(x_{t+1}|y_{1:t}) &= \int p(x_{t+1}|x_t)p(x_t|y_{1:t}) dx_t \\
 &\propto \int \exp \left\{ -\frac{1}{2\sigma_W^2}(x_{t+1} - \beta x_t)^2 \right\} \exp \left\{ -\frac{1}{2P_{t|t}}(x_t - a_{t|t})^2 \right\} dx_t \\
 &= \int \exp \left\{ -\frac{\beta^2}{2\sigma_W^2}(x_t - x_{t+1}/\beta)^2 - \frac{1}{2P_{t|t}}(x_t - a_{t|t})^2 \right\} dx_t \\
 &= \int \exp \left\{ -\frac{1}{2} \left[\frac{\beta^2}{\sigma_W^2}(x_t - x_{t+1}/\beta)^2 + \frac{1}{P_{t|t}}(x_t - a_{t|t})^2 \right] \right\} dx_t
 \end{aligned}$$

We thus have a complete-the-squares problem in the exponent in the integrand, with

$$A = \frac{\beta^2}{\sigma_W^2} \quad a = \frac{x_{t+1}}{\beta^2} \quad B = \frac{1}{P_{t|t}} \quad b = a_{t|t}$$

Therefore

$$\begin{aligned}
 p(x_{t+1}|y_{1:t}) &\propto \int \exp \left\{ -\frac{1}{2} \left[(A+B)(x_t - m)^2 + \frac{AB}{A+B}(a-b)^2 \right] \right\} dx_t \\
 &= \exp \left\{ -\frac{1}{2} \left[\frac{AB}{A+B}(a-b)^2 \right] \right\} \int \exp \left\{ -\frac{1}{2} [(A+B)(x_t - m)^2] \right\} dx_t \\
 &\propto \exp \left\{ -\frac{1}{2} \left[\frac{AB}{A+B}(a-b)^2 \right] \right\}
 \end{aligned}$$

as the integral does not depend on x_{t+1} (the quantity m depends on x_{t+1} , but the result of the integral does not).

The multiplicative constant that comes out in front of the integral is determined by the term

$$\frac{AB}{A+B}(a-b)^2 = \frac{\frac{\beta^2}{\sigma_W^2} \frac{1}{P_{t|t}}}{\frac{\beta^2}{\sigma_W^2} + \frac{1}{P_{t|t}}} (x_{t+1}/\beta - a_{t|t})^2 = \frac{1}{P_{t|t}\beta^2 + \sigma_W^2} (x_{t+1} - \beta a_{t|t})^2$$

from which we determine that

$$p(x_{t+1}|y_{1:t}) \equiv \mathcal{N}(\beta a_{t|t}, P_{t|t}\beta^2 + \sigma_W^2) \equiv \mathcal{N}(a_{t+1|t}, P_{t+1|t})$$

say. This can also be deduced from the iterated expectation and iterated variance results. We initialize the recursion with $a_{1|0} = x_1$ and $P_{1|0} = \sigma_V^2$.

Q5a

Note that we can deduce the above forms for the prediction densities using iterated expectation: we know that the density is Normal, so can deduce the distribution by computing the expectation and variance. That is, we have that

$$p(x_{t+1}|y_{1:t}) = \int p(x_{t+1}|x_t)p(x_t|y_{1:t}) dx_t.$$

but taking expectations in the state equation, we have that

$$a_{t+1|t} = \mathbb{E}_{X_t|Y_{1:t}}[X_{t+1}|y_{1:t}] = \mathbb{E}_{X_t|Y_{1:t}}[\beta X_t + W_t|y_{1:t}] = \beta \mathbb{E}_{X_t|Y_{1:t}}[X_t|y_{1:t}] = \beta a_{t|t}$$

and

$$P_{t+1|t} = \text{Var}_{X_t|Y_{1:t}}[\beta X_t + W_t|y_{1:t}] = \beta^2 \text{Var}_{X_t|Y_{1:t}}[X_t|y_{1:t}] + \sigma_W^2 = \beta^2 P_{t|t} + \sigma_W^2.$$

For filtering, we have

$$\begin{aligned} p(x_t|y_{1:t}) &\propto p(y_t|x_t)p(x_t|y_{1:t-1}) \\ &\propto \exp\left\{-\frac{1}{2\sigma_V^2}(y_t - x_t)^2\right\} \exp\left\{-\frac{1}{2P_{t|t-1}}(x_t - a_{t|t-1})^2\right\} \\ &\propto \exp\left\{-\frac{1}{2P_{t|t}}(x_t - a_{t|t})^2\right\} dx_t \end{aligned}$$

where, using the complete-the-square formula,

$$a_{t|t} = \frac{y_t/\sigma_V^2 + a_{t|t-1}/P_{t|t-1}}{1/\sigma_V^2 + 1/P_{t|t-1}} \quad P_{t|t}^{-1} = \frac{1}{\sigma_V^2} + \frac{1}{P_{t|t-1}}.$$

Hence

$$p(x_t|y_{1:t}) \equiv \mathcal{N}(a_{t|t}, P_{t|t}).$$

- (b) Explain how the densities in (c) facilitate computationally efficient likelihood-based estimation of the system parameters $\beta, \sigma_V, \sigma_W$. 6 MARKS

Q5a

We have in the filtering calculation

$$p(x_t|y_{1:t}) = \frac{p(y_t|x_t)p(x_t|y_{1:(t-1)})}{p(y_t|y_{1:(t-1)})} = \frac{p(y_t|x_t)p(x_t|y_{1:(t-1)})}{\int p(y_t|x_t^*)p(x_t^*|y_{1:(t-1)}) dx_t^*}$$

and the denominator in this calculation is precisely what we need for the likelihood. From the observation equation, we have that

$$\mathbb{E}_{Y_{t+1}|Y_{1:t}}[Y_{t+1}|y_{1:t}] = \mathbb{E}_{Y_{t+1}|Y_{1:t}}[X_{t+1} + V_{t+1}|y_{1:t}] = a_{t+1|t}$$

and

$$\text{Var}_{Y_{t+1}|Y_{1:t}}[Y_{t+1}|y_{1:t}] = \text{Var}_{Y_{t+1}|Y_{1:t}}[X_{t+1} + V_{t+1}|y_{1:t}] = \text{Var}_{Y_{t+1}|Y_{1:t}}[X_{t+1}|y_{1:t}] + \sigma_W^2 = P_{t+1|t} + \sigma_W^2$$

and so

$$p(y_{t+1}|y_{1:t}) \equiv \text{Normal}(a_{t+1|t}, P_{t+1|t} + \sigma_W^2)$$

and we can compute the likelihood by using this recursive formula

$$p(y_1) \prod_{t=1}^{n-1} p(y_{t+1}|y_{1:t})$$

where

$$p(y_1) = \int p(y_1|x_1)p(x_1) dx_1$$

can also be computed to be Normal.

Once we have the likelihood numerically evaluated, we can perform optimization to get the maximum likelihood estimates.

- (c) Is $\{Y_t\}$ a stationary process? Justify your answer by checking the conditions for stationarity. 6 MARKS

Q5a

We have that $E[Y_t] = E[X_t] = 0$, and that

$$\mathbb{E}[Y_t Y_{t+h}] = \mathbb{E}[(X_t + V_t)(X_{t+h} + V_{t+h})] = \mathbb{E}[X_t X_{t+h}] + \mathbb{E}[V_t V_{t+h}]$$

by independence. Therefore

$$\gamma_Y(h) = \begin{cases} \gamma_X(0) + \sigma_V^2 & h = 0 \\ \gamma_X(h) & h = \pm 1, \pm 2, \dots \end{cases}$$

provided that $\{X_t\}$ is stationary, that is, provided $|\beta| \neq 1$. If $|\beta| = 1$, then $\{X_t\}$ is non-stationary, and therefore $\{Y_t\}$ is non-stationary.