



MATH 545: Introduction to Time Series Analysis

Practice Final
Winter 2025

First name (please write as legibly as possible within the boxes)

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Last name

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Student ID number

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MATH 545: Specific Instructions.

Please write your answers on the exam in the boxes provided for each question; only answers written inside the correct boxes will be marked. Extra space for answers can be found at the end of the exam paper. Please write clearly.

This paper contains five questions. Each question carries 20 marks. Credit will be given for all questions attempted. The total mark available is 100 but rescaling of the final mark may occur.

Throughout the exam, $\{Z_t\}$ refers to a white noise process with variance σ_Z^2 unless otherwise stated; when stationarity is referred to, it is weak stationarity.

Only non-programmable and non text-storing calculators may be used. Dictionaries and translation dictionaries are permitted. A formula sheet is provided.



1. (a) What is meant by saying that a stochastic process is stationary?

2 MARKS

Q1a

(b) For an $AR(p)$ process defined by the equation

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t$$

what is the condition on the coefficients $\phi_1, \dots, \phi_p$ that is required for the process to be stationary?

4 MARKS

Q1b



(c) Consider the following AR(2) model,

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t.$$

that is known to be stationary.

(i) Using the backward shift operator B , show that this model can be written in the infinite MA or linear process form

$$X_t = \sum_{j=0}^{\infty} \frac{\xi_1^{j+1} - \xi_2^{j+1}}{\xi_1 - \xi_2} Z_{t-j},$$

where

$$(1 - \phi_1 B - \phi_2 B^2) = (1 - \xi_1 B)(1 - \xi_2 B).$$

8 MARKS

Q1ci



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Q1ci



- (ii) Use the expression derived in part (c) (i) to verify that the condition given in part (b) does indeed imply stationarity for $\{X_t\}$. 6 MARKS

Q1cii



2. The process $\{X_t\}$ is given by,

$$X_t = \alpha + \beta t + Y_t,$$

where $\{Y_t\}$ is a stationary process.

- (a) Show that differencing $\{X_t\}$ twice can be used to produce a new process $\{X_t^*\}$

$$X_t^* = \nabla^2 X_t = (1 - B)^2 X_t,$$

which does not depend on α or β . Determine whether $\{X_t^*\}$ is stationary. 6 MARKS

Q2a



- (b) (i) What property/properties of a linear filter, $\Psi(B)$ with coefficients $\{\psi_j\}$ are required for the process $\{Y_t\}$,

$$Y_t = \Psi(B)Z_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$$

to be well-defined in *mean-square* terms ? Give the definition of mean-square equality of random variables. 4 MARKS

Q2bi

- (ii) If $\{U_t\}$ defined by $U_t = \Psi(B)Z_t$, derive the general form of the autocorrelation function (ACF), $\rho_U(h)$, in terms of the coefficients of $\Psi(B)$. 4 MARKS

Q2bii



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(iii) If $\{U_t\}$ is an MA(2) process given by,

$$U_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}.$$

Determine the ACVF for $\{U_t\}$ in terms of θ_1, θ_2 and σ_Z^2 .

6 MARKS

Q2c



3. (a) A stationary stochastic process is termed *invertible* if it admits a *causal* $AR(\infty)$ representation.

(i) Define the term causal used in this definition.

2 MARKS

Q3ai

(ii) Show that any stationary $AR(p)$ process is invertible.

3 MARKS

Q3aii

(iii) Write down a condition, involving the roots or reciprocal roots of an equation, for an $MA(q)$ process to be invertible.

3 MARKS

Q3aiii



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- (b) (i) Determine the autocovariance function (ACVF), $\gamma_X(h)$, $h = 0, \pm 1, \pm 2, \dots$ for the MA(1) process,

$$X_t = Z_t + \theta_1 Z_{t-1}.$$

4 MARKS

Q3bi



- (ii) When modelling a process $\{X_t\}$ as an MA(1) process, show that there is more than one such process with $\gamma_X(0) = 5.0$ and $\gamma_X(1) = 2.0$. Verify that only one of these processes is invertible. 8 MARKS

Q3bii



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4. (a) Explain the difference between *deterministic* and *stochastic* seasonal behaviour, and describe a model or method designed to handle each type of behaviour. 6 MARKS

Q4a



(b) Write down the AR and/or MA polynomials for the following SARIMA models, which are written in the usual notation as

(i) $\text{SARIMA}(0, 1, 1) \times (1, 1, 0)_{12}$.

3 MARKS

(ii) $\text{SARIMA}(1, 0, 0) \times (1, 0, 1)_4$.

3 MARKS

(iii) $\text{SARIMA}(0, 1, 0) \times (1, 0, 0)_7$.

3 MARKS

Q4b

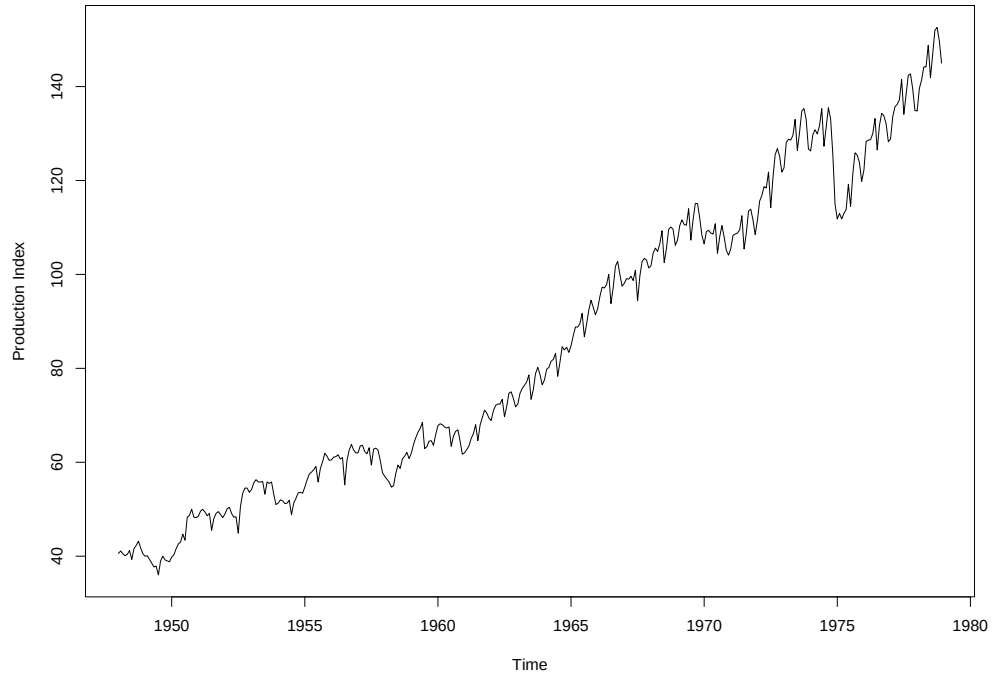


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- (c) The plot below depicts monthly US Federal Reserve Board production index values, 1948-1978 ($n = 372$ months).



- (i) Is this a realization of a stationary process ? Justify your answer. 2 MARKS

Q4ci



- (ii) Give one suggestion for a data transformation that may make the resulting transformed series stationary ? 2 MARKS

Q4cii

- (iii) Outline an analysis strategy that you would use to analyze these data if you were attempting to construct a realistic model for forecasting. 4 MARKS

Q4ciii



5. Consider the following state space model

$$Y_t = X_t + V_t$$

$$X_{t+1} = \beta X_t + W_t$$

where $\{V_t\}$ and $\{W_t\}$ are independent, zero-mean Gaussian IID processes with variances σ_V^2 and σ_W^2 respectively, and β is a real-valued parameter.

(a) Derive the recursion for the filtering and prediction densities

$$p(x_t|y_{1:t}) \quad p(x_{t+1}|y_{1:t})$$

in this model, that is, show that these distributions are Normal, with means and variances determined by the means and variances of the preceding densities

$$p(x_{t-1}|y_{1:(t-1)}) \quad p(x_t|y_{1:(t-1)})$$

which are also Normal.

8 MARKS

Hint: the univariate Gaussian density with mean m and variance v takes the form

$$f(x|m, v) = \left(\frac{1}{2\pi v}\right)^{1/2} \exp\left\{-\frac{1}{2v}(x - m)^2\right\}.$$

so that, for example, in the model in (c),

$$f(x_{t+1}|x_t) = \left(\frac{1}{2\pi\sigma_W^2}\right)^{1/2} \exp\left\{-\frac{1}{2\sigma_W^2}(x_{t+1} - \beta x_t)^2\right\}.$$

and

$$f(y_t|x_t) = \left(\frac{1}{2\pi\sigma_V^2}\right)^{1/2} \exp\left\{-\frac{1}{2\sigma_V^2}(y_t - x_t)^2\right\}.$$

The following completing-the-square identity may be useful:

$$A(x - a)^2 + B(x - b)^2 = (A + B)(x - m)^2 + \frac{AB}{A + B}(a - b)^2$$

where

$$m = \frac{Aa + Bb}{A + B}.$$



Q5a



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- (b) Explain how the densities in (c) facilitate computationally efficient likelihood-based estimation of the system parameters $\beta, \sigma_W, \sigma_V$. 6 MARKS

Q5a



- (c) Is $\{Y_t\}$ a stationary process? Justify your answer by checking the conditions for stationarity. 6 MARKS

Q5a



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