

MATH 545: Introduction to Time Series Analysis

Final Examination: Solutions

Winter 2025

April 29th 2025: 9.00am – 12.00pm

Note that these solutions are not the only way to get full or part marks. The grading has been completed allowing for variants of the solution.

MATH 545: Specific Instructions.

Please write your answers on the exam in the boxes provided for each question; only answers written inside the correct boxes will be marked. Extra space for answers can be found at the end of the exam paper. Please write clearly.

This paper contains five questions. Each question carries 20 marks. Credit will be given for all questions attempted. The total mark available is 100 but rescaling of the final mark may occur.

Throughout the exam, $\{Z_t\}$ refers to a white noise process with variance σ_Z^2 unless otherwise stated; when stationarity is referred to, it is weak stationarity.

Only non-programmable and non text-storing calculators may be used. Dictionaries and translation dictionaries are permitted. A formula sheet is provided.

1. (a) Show that the process $\{X_t\}$ defined as the solution to the equation

$$X_t = X_{t-1} + Z_t$$

is a non-stationary process.

4 MARKS

Q1(a)

This is the random walk process, which is non-stationary unless $\sigma_Z^2 = 0$, as it is evident that for any process satisfying this equation,

$$\text{Var}[X_t] \geq \text{Var}[X_{t-1}]$$

with the inequality being strict unless $\sigma_Z^2 = 0$, in which case the process is almost surely constant.

Alternatively, could note that this is an AR(1) process with $\phi = 1$, that is a unit root, so we can establish that no stationary solution exists; by the recursive approach we have

$$X_t = \sum_{j=0}^{\infty} Z_{t-j}$$

and the partial sums defined as

$$\sum_{j=0}^N Z_{t-j}$$

for $N = 1, 2, \dots$ diverge almost surely.

(b) Which of the following processes $\{X_t\}$ are stationary ? Give reasoning for your answer.

(i) $X_t = Z_t + Z_{t-1}^2$. 4 MARKS

(ii) $X_t = Z_t Z_{t-1}$. 4 MARKS

(iii) $X_t = 2X_{t-1} + X_{t-2} + Z_t$. 4 MARKS

Q1(b)

(i) This process is stationary provided $\mathbb{E}[Z_t^4]$ is finite, as it is a q -dependent process with $q = 1$. We have $\mathbb{E}[X_t] = \mathbb{E}[Z_t^2] = \sigma_Z^2$ and

$$\text{Var}[X_t] = \text{Var}[Z_t] + \text{Var}[Z_t^2] = \sigma_Z^2 + \mathbb{E}[Z_t^4] - \mathbb{E}[Z_t^2]^2 = \sigma_Z^2 + \mathbb{E}[Z_t^4] - \sigma_Z^4.$$

and

$$\mathbb{E}[X_t X_{t-1}] = \mathbb{E}[(Z_t + Z_{t-1}^2)(Z_{t-1} + Z_{t-2}^2)] = \mathbb{E}[Z_t^3] + \mathbb{E}[Z_t^2]^2 = \mathbb{E}[Z_t^3] + \sigma_Z^4$$

so $\gamma_X(1) = \mathbb{E}[Z_t^3]$. For $h \geq 2$, $\mathbb{E}[X_t X_{t-h}] = \mathbb{E}[Z_t^2]^2 = \sigma_Z^4$, so $\gamma_X(h) = 0$. 4 MARKS

(ii) This process is stationary as it is a q -dependent process with $q = 1$. Further we have $\mathbb{E}[X_t] = 0$, and

$$\text{Var}[X_t] = \mathbb{E}[X_t^2] = \mathbb{E}[Z_t^2 Z_{t-1}^2] = \mathbb{E}[Z_t^2] \mathbb{E}[Z_{t-1}^2] = \sigma_Z^4.$$

For $h = 1$

$$\text{Cov}[X_t, X_{t-1}] = \mathbb{E}[X_t X_{t-1}] = \mathbb{E}[Z_t Z_{t-1} Z_{t-1} Z_{t-2}] = 0$$

by uncorrelatedness. A similar expansion holds for $h \geq 2$. Therefore

$$\gamma_X(h) = \begin{cases} \sigma_Z^4 & h = 0 \\ 0 & h \geq 1 \end{cases}$$

4 MARKS

(iii) Here we have the AR polynomial

$$\Phi(z) = 1 - 2z - z^2 = -(z^2 + 2z - 1).$$

The roots of this polynomial are

$$(\xi_1, \xi_2) = \frac{-2 \pm \sqrt{4 + 4}}{2} = -1 \pm \sqrt{2} = (-1 - \sqrt{2}, -1 + \sqrt{2})$$

Neither of the roots (or reciprocal roots) is 1 in modulus, therefore the process is stationary, however, the process is not causal, as one of the reciprocal roots has modulus greater than 1. 4 MARKS

(c) Suppose that $\{Y_t\}$ is defined by

$$Y_t = \exp\{X_t/2\}Z_t$$

where $\{Z_t\}$ is a white noise process with variance $\sigma_Z^2 = 1$, and $\{X_t\}$ satisfies

$$X_t = \phi X_{t-1} + W_t$$

where $|\phi| < 1$, with $\{W_t\}$ a white noise process with variance σ_W^2 . Find the autocovariance function (ACVF) for $\{Y_t\}$, $\gamma_Y(h)$. **4 MARKS**

You may assume that $\{Z_t\}$ and $\{W_t\}$ are uncorrelated.

Q1(c)

The conditions ensure that $\{X_t\}$ is stationary, and therefore that the process

$$\{\exp\{X_t/2\}, t \in \mathbb{Z}\}$$

is stationary provided $\mathbb{E}[\exp\{X_t\}] < \infty$. Under this assumption, we have that

$$\mathbb{E}[Y_t] = \mathbb{E}[\exp\{X_t/2\}Z_t] = \mathbb{E}[\exp\{X_t/2\}]\mathbb{E}[Z_t] = 0$$

as $\mathbb{E}[Z_t] = 0$, and for $h > 0$

$$\begin{aligned}\gamma_Y(h) &= \mathbb{E}[Y_t Y_{t-h}] \\ &= \mathbb{E}[\exp\{X_t/2\}Z_t \exp\{X_{t-h}/2\}Z_{t-h}] \\ &= \mathbb{E}[\exp\{(X_t + X_{t-h})/2\}]\mathbb{E}[Z_t Z_{t-h}] = 0\end{aligned}$$

as $\{Z_t\}$ is a white noise process, and $\mathbb{E}[\exp\{(X_t + X_{t-h})/2\}]$ is finite by Cauchy-Schwarz.

For $h = 0$

$$\gamma_Y(0) = \text{Var}[Y_t] = \mathbb{E}[Y_t^2] = \mathbb{E}[\exp\{X_t\}Z_t^2] = \mathbb{E}[\exp\{X_t\}]$$

as $\mathbb{E}[Z_t^2] = 1$.

2. The linear process representation of $\{X_t\}$ takes the form

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$$

where it is assumed that $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ to ensure that $\{X_t\}$ is well-defined.

(a) Derive an expression for $\text{Var}[X_t]$ in terms of ψ_j , $j = 0, \pm 1, \pm 2, \dots$

6 MARKS

Q2(a)

By direct calculation, due to uncorrelatedness

$$\text{Var}[Y_t] = \sum_{j=-\infty}^{\infty} \psi_j^2 \text{Var}[Z_{t-j}] = \sigma_Z^2 \sum_{j=-\infty}^{\infty} \psi_j^2$$

(b) What condition on the ψ_j is required for $\{X_t\}$ to be a *causal* process ?

2 MARKS

Q2(b)

We need $\psi_j = 0$ for $j < 0$.

(c) Suppose that $\{X_t\}$ is defined by the equation

$$(1 - \phi B)X_t = (1 + \theta B)Z_t,$$

for parameters ϕ and θ , with $|\phi| < 1$, and that $\{Y_t\}$ is defined by $Y_t = (1 - B)X_t$.

Find the linear process representation for $\{Y_t\}$. If no such representation exists, explain why. 8 MARKS

Q2(c)

We have that

$$X_t = (1 - \phi B)^{-1}(1 + \theta B)Z_t$$

so therefore

$$Y_t = (1 - B)X_t = (1 - B)(1 - \phi B)^{-1}(1 + \theta B)Z_t = (1 - \phi B)^{-1}(1 - B)(1 + \theta B)Z_t$$

and hence

$$(1 - \phi B)Y_t = (1 - B)(1 + \theta B)Z_t$$

so $\{Y_t\}$ is a stationary and causal ARMA(1,2). We note that the linear process representation is given by the polynomial

$$\begin{aligned}\Psi(z) &= (1 - \phi z)^{-1}(1 - z)(1 + \theta z) \\ &= (1 + \phi z + \phi^2 z^2 + \phi^3 z^3 \dots)(1 + (\theta - 1)z - \theta z^2)\end{aligned}$$

and hence by computing coefficients we have

$$\psi_0 = 1$$

$$\psi_1 = \phi + \theta - 1$$

$$\psi_2 = \phi^2 - \theta + \phi(\theta - 1)$$

$$\psi_3 = \phi^3 - \phi\theta + \phi^2(\theta - 1) = \phi\psi_2$$

etc, and so for $j \geq 3$, $\psi_j = \phi^{j-2}\psi_2$.

- (d) Suppose that $\{X_{1,t}\}$ is an MA(1) process and $\{X_{2,t}\}$ is an MA(2) process, with the two processes independent. For all t , let

$$X_t = X_{1,t} + X_{2,t}.$$

Does $\{X_t\}$ have a linear process representation? Justify your answer. 4 MARKS

Q2(c)

Suppose that

$$X_{1,t} = Z_{1,t} + \theta_{1,1}Z_{1,t-1} = \Theta_1(B)Z_{1,t}$$

and

$$X_{2,t} = Z_{2,t} + \theta_{2,1}Z_{2,t-1} + \theta_{2,2}Z_{2,t-2} = \Theta_2(B)Z_{2,t}.$$

By properties of MA processes, we have that

$$\gamma_{X_1}(h) = \begin{cases} \sigma_1^2(1 + \theta_{1,1}^2) & h = 0 \\ \sigma_1^2\theta_{1,1} & h = \pm 1 \\ 0 & |h| > 2 \end{cases} \quad \gamma_{X_2}(h) = \begin{cases} \sigma_2^2(1 + \theta_{2,1}^2 + \theta_{2,2}^2) & h = 0 \\ \sigma_2^2(\theta_{2,1} + \theta_{2,1}\theta_{2,2}) & h = \pm 1 \\ \sigma_2^2\theta_{2,2} & h = \pm 2 \\ 0 & |h| > 3 \end{cases}$$

say. Now by independence

$$\gamma_X(h) = \gamma_{X_1}(h) + \gamma_{X_2}(h)$$

and clearly $\gamma_X(h)$ is non-negative definite, so therefore we have that $\{X_t\}$ has the same ACVF as an MA(2) process. Therefore $\{X_t\}$ does have a linear process representation; that is, there exists a white noise process, $\{Z_t\}$, and coefficients θ_1 and θ_2 such that

$$X_t \stackrel{\text{m.s.}}{=} Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}$$

More formally, consider the autocovariance generating function (ACVGF) of $\{X_t\}$; we have that

$$G_X(z) = \sum_{h=-\infty}^{\infty} \gamma_X(h)z^h = \sum_{h=-\infty}^{\infty} (\gamma_{X_1}(h) + \gamma_{X_2}(h))z^h = G_1(z) + G_2(z)$$

But from above

$$G_1(z) + G_2(z) = \sum_{h=-2}^2 (c_1(h) + c_2(h))z^h$$

say. To find the corresponding MA polynomial, $\Theta(z)$, we need to equate

$$\sigma_Z^2 \Theta(z) \Theta(z^{-1}) = \sigma_1^2 \Theta_1(z) \Theta_1(z^{-1}) + \sigma_2^2 \Theta_2(z) \Theta_2(z^{-1})$$

3. Consider the stationary and causal AR(2) model

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t.$$

- (a) Derive a system of linear equations and recursion to determine the autocovariance function (ACVF), $\gamma_X(h)$. You do not need to solve this system. 8 MARKS

Q3(a)

We have by the usual direct calculation method for this stationary and causal case

$$\mathbb{E}[X_t X_{t-h}] = \phi_1 \mathbb{E}[X_{t-1} X_{t-h}] + \phi_2 \mathbb{E}[X_{t-2} X_{t-h}] + \mathbb{E}[Z_t X_{t-h}]$$

so that

$$\begin{aligned} h = 0 & \quad \gamma_X(0) = \phi_1 \gamma_X(1) + \phi_2 \gamma_X(2) + \sigma_Z^2 \\ h = 1 & \quad \gamma_X(1) = \phi_1 \gamma_X(0) + \phi_2 \gamma_X(1) \\ h = 2 & \quad \gamma_X(2) = \phi_1 \gamma_X(1) + \phi_2 \gamma_X(0) \end{aligned}$$

and for $h \geq 3$, we have the recursion

$$\gamma_X(h) = \phi_1 \gamma_X(h-1) + \phi_2 \gamma_X(h-2)$$

The first three equations determine a linear system.

- (b) Describe, giving mathematical details, one method that could be used to estimate ϕ_1 and ϕ_2 from sample data. 4 MARKS

Q3(b)

The simplest method for AR(2) is OLS where we minimize

$$\sum_{t=3}^n (x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2})^2$$

with respect to ϕ_1 and ϕ_2 . This is equivalent to regressing X_t on (X_{t-1}, X_{t-2}) . Other methods include

- Yule-Walker (solving the system in the previous part)
- Levinson-Durbin to compute PACFs, which in this case are precisely the ϕ s.
- PACFs (Burg's method)
- Maximum likelihood (if we are happy to make distributional assumptions)

(c) Suppose that for $t = 1, \dots, n$,

$$Y_t = \beta_0 + \beta_1 t + X_t$$

for constants β_0 and β_1 , where $\{X_t\}$ is a stationary and causal ARMA(p, q) process.

Describe how to produce the optimal linear one-step-ahead forecast, $\hat{Y}_{n+1}$, based on the mean-square prediction error criterion. 8 MARKS

Q3(c)

The trick here is to note that

$$\hat{Y}_{n+1} = \beta_0 + \beta_1(n+1) + \hat{X}_{n+1}$$

where $\hat{X}_{n+1}$ is the optimal linear forecast for the $\{X_t\}$ process. We find $\hat{X}_{n+1}$ using the usual method: we note that

$$\hat{X}_{n+1} = \sum_{i=1}^n \hat{a}_i X_{n-i+1}$$

where $\hat{\mathbf{a}}$ solves the linear system

$$\mathbf{\Gamma} \mathbf{a} = \boldsymbol{\gamma}_{1:n}$$

where $\mathbf{\Gamma}$ is the Toeplitz autocovariance matrix of $(X_1, \dots, X_n)^\top$ with first row

$$[\gamma_X(0), \gamma_X(1), \dots, \gamma_X(n-1)]$$

and

$$\boldsymbol{\gamma}_{1:n} = [\gamma_X(1), \gamma_X(2), \dots, \gamma_X(n)]^\top.$$

We can solve the linear system directly, or using the Levinson-Durbin algorithm.

You could also use differencing, as

$$\nabla Y_t = \beta_1 + \nabla X_t$$

then forecast $\{\nabla X_t\}$ as a stationary ARMA process, then undo the differencing.

4. Suppose that $\{X_t\}$ satisfies

$$(1 - \phi_1 B^4)(1 - \phi_2 B)^2 X_t = (1 + \theta B^4) Z_t$$

for parameters ϕ_1, ϕ_2, θ , with $0 \leq \phi_1, \phi_2 \leq 1$. Answer, with reasoning, the following questions.

(a) If $\phi_2 = 1$ and $\phi_1 < 1$, and $Y_t = (1 - B)^2 X_t$, is $\{Y_t\}$ stationary ?

5 MARKS

Q4(a)

Here we have

$$(1 - \phi_1 B^4)(1 - B)^2 X_t = (1 + \theta B^4) Z_t$$

so that

$$(1 - \phi_1 B^4) Y_t = (1 + \theta B^4) Z_t$$

so $\{Y_t\}$ is a stationary process: it is a SARIMA(0, 0, 0) $\times$ (1, 0, 1)₄ process.

(b) If $\phi_2 = 1$ and $\phi_1 < 1$, and $Y_t = (1 - B^4)(1 - B)^2 X_t$, is $\{Y_t\}$ stationary ?

5 MARKS

Q4(b)

We know from (a) that if $W_t = (1 - B)^2 X_t$, then

$$(1 - \phi_1 B^4) W_t = (1 + \theta B^4) Z_t$$

and $\{W_t\}$ is stationary. Now $Y_t = (1 - B^4) W_t$. This is a seasonal first difference of a stationary process, so therefore $\{Y_t\}$ is stationary also.

- (c) Suppose that $\theta = 0$ and $\phi_1, \phi_2 < 1$. Find the autocovariance generating function (ACVGF) for $\{X_t\}$. 5 MARKS

Q4(c)

Here we have that

$$(1 - \phi_1 B^4)(1 - \phi_2 B)^2 X_t = Z_t$$

Using the formula sheet, we have that the ACVGF is

$$G_X(z) = \frac{\sigma_Z^2}{(1 - \phi_1 z^4)(1 - \phi_2 z)^2(1 - \phi_1 z^{-4})(1 - \phi_2 z^{-1})^2}$$

- (d) If $\phi_2 = 0$ and $\phi_1 < 1$, compute first three *non-zero* coefficients of the causal linear process representation 5 MARKS

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}.$$

Q4(d)

Here we have

$$(1 - \phi_1 B^4)X_t = (1 + \theta B^4)Z_t$$

so that

$$\begin{aligned} X_t &= (1 - \phi_1 B^4)^{-1}(1 + \theta B^4)Z_t \\ &= (1 + \phi_1 B^4 + \phi_1^2 B^8 + \cdots)(1 + \theta B^4)Z_t \\ &= 1 + (\phi_1 + \theta)B^4 Z_t + \phi_1(\phi_1 + \theta)B^8 Z_t + \cdots \\ &= 1 + (\phi_1 + \theta)Z_{t-4} + \phi_1(\phi_1 + \theta)Z_{t-8} + \cdots \end{aligned}$$

5. (a) Find the linear state space representation for the stationary and causal AR(3) model

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 X_{t-3} + Z_t.$$

6 MARKS

Q5(a)

From the formula sheet, for ARMA(p, q) models: let $m = \max\{p, q + 1\}$, and

$$\begin{aligned} Y_t &= \mathbf{F}_t \mathbf{X}_t + V_t \\ \mathbf{X}_{t+1} &= \mathbf{G}_t \mathbf{X}_t + \mathbf{W}_t \end{aligned}$$

with state $\mathbf{X}_t = (X_{1,t-m+1}, X_{1,t-m+2}, \dots, X_{1,t})^\top$, and $\mathbf{W}_t = (0, 0, 0, \dots, Z_{t+1})^\top$, so that

$$\mathbf{G}_t = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \phi_m & \phi_{m-1} & \phi_{m-2} & \cdots & \phi_2 & \phi_1 \end{bmatrix} \quad \mathbf{F}_t = [\theta_{m-1} \quad \theta_{m-2} \quad \cdots \quad \theta_1 \quad \theta_0].$$

with $V_t \equiv 0$ and $Z_t \sim \text{Normal}(0, \sigma_Z^2)$.

Here, $m = 3$, and

$$\mathbf{X}_t = \begin{bmatrix} X_{1,t-2} \\ X_{1,t-1} \\ X_{1,t} \end{bmatrix} \quad \mathbf{G}_t = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \phi_3 & \phi_2 & \phi_1 \end{bmatrix} \quad \mathbf{F}_t = [0 \quad 0 \quad 1] \quad \mathbf{W}_t = \begin{bmatrix} 0 \\ 0 \\ Z_{t+1} \end{bmatrix}$$

with $V_t \equiv 0$ and $Z_t \sim \text{Normal}(0, \sigma_Z^2)$.

(b) Derive the autocorrelation function (ACF) of the process $\{Y_t\}$ if

$$Y_t = X_t + V_t$$

$$X_{t+1} = \phi X_t + W_t$$

where $|\phi| < 1$, and $\{V_t\}$ and $\{W_t\}$ are white noise processes with variances 1 and σ_W^2 respectively, with $\text{Cov}[V_t, W_s] = 0$ for all t, s . 8 MARKS

Q5(b)

Here $\{Y_t\}$ is clearly stationary as $\{X_t\}$ is stationary. We have that

$$\mathbb{E}[Y_t Y_{t-h}] = \mathbb{E}[(X_t + V_t)(X_{t-h} + V_{t-h})] = \gamma_X(h) + \gamma_V(h)$$

We have from first principles that for each h

$$\gamma_X(h) = \frac{\sigma_W^2}{1 - \phi^2} \phi^h$$

and $\gamma_V(h) = 1$ for $h = 0$ and zero otherwise. Therefore

$$\gamma_Y(h) = \begin{cases} \frac{\sigma_W^2}{1 - \phi^2} + 1 & h = 0 \\ \frac{\sigma_W^2}{1 - \phi^2} \phi^h & h = 1, 2, \dots \end{cases}$$

and hence the ACF is

$$\rho_Y(h) = \frac{\gamma_Y(h)}{\gamma_Y(0)} = \frac{\sigma_W^2 \phi^h}{\sigma_W^2 + (1 - \phi^2)}$$

for $h = 1, 2, \dots$

(c) Suppose

$$X_t = \sum_{j=1}^k [U_j \cos(\omega_j t) + V_j \sin(\omega_j t)]$$

where $U_1, \dots, U_k$ and $V_1, \dots, V_k$ are uncorrelated, with

$$\mathbb{E}[U_j] = \mathbb{E}[V_j] = 0 \quad \text{Var}[U_j] = \text{Var}[V_j] = \frac{1}{j^2}$$

and $0 < \omega_1 < \dots < \omega_k < \pi$ are fixed constants.

Find the autocovariance function (ACVF), $\gamma_X(h)$, for $\{X_t\}$

4 MARKS

Q5(c)

We have that for all t , and for each component j

$$\mathbb{E}[X_{j,t}] = \mathbb{E}[U_j \cos(\omega_j t)] + \mathbb{E}[V_j \sin(\omega_j t)] = 0$$

and by direct calculation

$$\begin{aligned} \text{Cov}[X_{j,t}, X_{j,t+h}] &= \mathbb{E}[X_{j,t} X_{j,t+h}] \\ &= \mathbb{E}[(U \cos(\omega_j t) + V \sin(\omega_j t))(U_j \cos(\omega_j(t+h)) + V_j \sin(\omega_j(t+h)))] \\ &= \mathbb{E}[U_j^2] \cos(\omega_j t) \cos(\omega_j(t+h)) + \mathbb{E}[V_j^2] \sin(\omega_j t) \sin(\omega_j(t+h)) \\ &= \frac{1}{j^2} \cos(\omega_j t) \cos(\omega_j(t+h)) + \frac{1}{j^2} \sin(\omega_j t) \sin(\omega_j(t+h)) \\ &= \cos(\omega_j(-h))/j^2 = \cos(\omega_j h)/j^2. \end{aligned}$$

By uncorrelatedness, we have that

$$\gamma_X(h) = \text{Cov}[X_t, X_{t+h}] = \sum_{j=1}^k \text{Cov}[X_{j,t}, X_{j,t+h}] = \sum_{j=1}^k \frac{1}{j^2} \cos(\omega_j h)$$

(d) Find the spectral distribution function, F_X , for the process $\{X_t\}$ in part (c).

2 MARKS

Q5(d)

We have from above that

$$\gamma_X(h) = \sum_{j=1}^k \gamma_{X_j}(h)$$

so therefore, using the definition,

$$F_X(\lambda) = \sum_{j=1}^k \frac{1}{j^2} F_{X_j}(\lambda)$$

where

$$F_{X_j}(\lambda) = \begin{cases} 0 & \lambda < -\omega_j \\ \frac{1}{2} & -\omega_j \leq \lambda < \omega_j \\ 1 & \omega_j \leq \lambda \end{cases}$$