

MATH 545
Key concepts

Basic Concepts

- time indexed random variables
- mean, variance, autocovariance
- stationarity: strong and weak stationarity
- autocovariance and autocorrelation

Elementary types of process

Basic properties of

- white noise process
- random walk

Simple stationary processes

- autoregressive processes (AR)
 - ▶ recursive definition for AR(1)
 - ▶ conditions for stationarity
 - ▶ mean-square convergence
 - ▶ mean, variance and autocovariance for AR(1)
 - ▶ AR(p) processes

Simple stationary processes (cont.)

- moving average processes (MA)
 - ▶ mean, variance and autocovariance
 - ▶ MA(1) processes
 - ▶ MA(q) processes

Simple stationary processes (cont.)

- linear processes
 - ▶ the general linear process representation
 - ▶ condition for a well-defined process
 - ▶ linear filters
 - ▶ mean-square convergence
 - ▶ causal processes
- predictable processes
 - ▶ harmonic processes

- backshift operator: $BX_t = X_{t-1}$
- difference operator: $\nabla X_t = (1 - B)X_t = X_t - X_{t-1}$
- seasonal difference operator: $\nabla_d X_t = (1 - B^d)X_t = X_t - X_{t-d}$
- polynomials in B eg

$$\Psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \cdots + \psi_j B^j + \cdots$$

Trends and Nonstationarity

- non constant mean
- deterministic seasonality
- decompositions

Key techniques

- to compute the ACVF from the representation:

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$$

then

$$\begin{aligned}\mathbb{E}[X_t X_{t+h}] &= \mathbb{E} \left[\left(\sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} \right) \left(\sum_{k=-\infty}^{\infty} \psi_k Z_{t+h-k} \right) \right] \\ &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \mathbb{E} [Z_{t-j} Z_{t+h-k}] \\ &= \sigma_Z^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h}\end{aligned}$$

Key techniques (cont.)

- direct computation: eg if

$$X_t = \phi X_{t-1} + Z_t$$

then

$$X_t X_{t-h} = \phi X_{t-1} X_{t-h} + Z_t X_{t-h}$$

so that

$$\mathbb{E}[X_t X_{t-h}] = \phi \mathbb{E}[X_{t-1} X_{t-h}] + \mathbb{E}[Z_t X_{t-h}]$$

and hence

$$\gamma_X(h) = \begin{cases} \phi \gamma_X(h-1) & h \geq 1 \\ \phi \gamma_X(1) + \sigma_Z^2 & h = 0 \end{cases}.$$

Also by direct computation for stationary processes

$$\mathbb{V}\text{ar}[X_t] = \phi^2 \mathbb{V}\text{ar}[X_{t-1}] + \mathbb{V}\text{ar}[Z_t]$$

so that

$$\gamma_X(0) = \phi^2 \gamma_X(0) + \sigma_Z^2$$

Key techniques (cont.)

- manipulating filter polynomials eg

$$\Phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p = \prod_{j=1}^p (1 - \xi_j z)$$

or

$$(1 - \xi z)^{-1} = 1 + \xi z + \xi^2 z^2 + \dots$$

- computing the linear process representation coefficients by expansion
- combining filters

Reciprocal roots

- Stationarity for AR processes
- Invertibility for MA processes

- optimal forecasts under minimum MSE
- optimal linear forecasts & conditional expectation solution
- solving the linear system
- forecast properties (expectation, variance, minimized MSE)
- Levinson-Durbin and innovations algorithm
- predictable processes

Partial autocorrelation

- definition
- relation to Levinson-Durbin

- definition

$$\Phi(B)X_t = \Theta(B)Z_t$$

or

$$X_t = \phi_1 X_{t-1} + \cdots \phi_p X_{t-p} + Z_t + \theta_1 Z_{t-1} + \cdots \theta_q Z_{t-q}$$

- properties
- ACF/PACF
- eg ARMA(1,1) properties

- manipulating the complex-valued polynomials

$$\Phi(z) = \prod_{j=1}^p (1 - \xi_j B)$$

$$\Theta(z) = \prod_{j=1}^q (1 - \omega_j B)$$

and

$$\Psi(z) = \frac{\Theta(z)}{\Phi(z)}$$

ARMA Processes (cont.)

- Direct computation via solving linear system and recursion for ARMA(p, q):

$$X_t = \sum_{j=1}^p \phi_j X_{t-j} + \sum_{j=0}^q \theta_j Z_{t-j}$$

then multiply through by X_{t-h} and take expectations

$$\gamma_X(h) = \sum_{j=1}^p \phi_j \gamma_X(h-j) + \sigma_Z^2 \sum_{j=0}^q \theta_j \psi_{j-h}$$

or equivalently

$$\gamma_X(h) = \sum_{j=1}^p \phi_j \gamma_X(h-j) + \sigma_Z^2 \sum_{j=0}^{q-h} \theta_{j+h} \psi_j.$$

If $m = \max\{p, q + 1\}$, this yields a linear system of $m - 1$ equations, after which

$$\gamma_X(h) = \sum_{j=1}^p \phi_j \gamma_X(h - j) \quad h = m, m + 1, \dots$$

and a straightforward computation.

Autocovariance Generating Function

- definition
- uses
 - ▶ identification of processes
 - ▶ constructing causal and invertible ARMA solutions
- form for ARMA(p, q)

$$G_Z(z) = \sigma_Z^2 \Psi(z) \Psi(z^{-1}) = \sigma_Z^2 \frac{\Theta(z)}{\Phi(z)} \frac{\Theta(z^{-1})}{\Phi(z^{-1})}$$

- model-free (moment-based) estimation of mean and ACVF
- methods for $AR(p)$
- maximum likelihood estimation
- using the innovations algorithm for Gaussian estimation
- properties of estimators

- residual checking
- model selection AIC/BIC etc.

- basic ARIMA formulation
- integrated (ie difference stationary) processes
- unit roots
- Dickey-Fuller testing

- seasonal ARIMA
- general analysis strategy
 - ▶ pre-processing (deterministic non-stationarity)
 - ▶ differencing (stochastic non-stationarity)
 - ▶ estimation
 - ▶ model selection
 - ▶ model checking

Vector-valued processes

- general formulation
- stationarity and ACVF/ACF calculations
- VAR/VMA/VARMA models
- co-integration and error-correction models

State space models

- general formulation
- linear and Gaussian state space models
- representations for non-stationary/dynamic linear models
- representations for ARMA models
- filtering/prediction/smoothing calculations

$$p(x_{t-1}|y_{1:(t-1)}), p(x_t|y_{1:(t-1)}) \longrightarrow p(x_t|y_{1:t}), p(x_{t+1}|y_{1:t})$$

- the Kalman filter
- hidden Markov models

Frequency domain modelling

- frequency representations of stationary series
- spectral density/distribution functions
- connection between ACVF and SDF
- connection between harmonic processes and SDF
- periodogram
- Whittle likelihood
- long memory models/fractional white noise

- observation-driven/parameter-driven models
- conditional heteroscedastic models
- ARCH/GARCH
- stochastic volatility