

MATH 545: INTRODUCTION TO TIME SERIES ANALYSIS

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NON-NEGATIVE DEFINITENESS OF COSINE

Suppose U and V are zero mean and uncorrelated random variables with variance 1. Let $\{X_t\}$ be defined by

$$X_t = U \cos(\omega t) + V \sin(\omega t)$$

where ω is a fixed constant. Then $\mathbb{E}[X_t] = 0$ for all t , and by direct calculation

$$\begin{aligned} \text{Cov}[X_t, X_{t+h}] &= \mathbb{E}[X_t X_{t+h}] \\ &= \mathbb{E}[(U \cos(\omega t) + V \sin(\omega t))(U \cos(\omega(t+h)) + V \sin(\omega(t+h)))] \\ &= \mathbb{E}[U^2] \cos(\omega t) \cos(\omega(t+h)) + \mathbb{E}[V^2] \sin(\omega t) \sin(\omega(t+h)) \\ &= \cos(\omega t) \cos(\omega(t+h)) + \sin(\omega t) \sin(\omega(t+h)) \\ &= \cos(\omega(-h)) \quad \text{as } \cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B) \\ &= \cos(\omega h) \end{aligned}$$

To demonstrate explicitly that this is a non-negative definite function, we need to demonstrate that for all $d \geq 1$ and vectors $(h_1, \dots, h_d)^\top \in \mathbb{Z}^d$

$$\sum_{i=1}^d \sum_{j=1}^d h_i h_j \cos(\omega(h_i - h_j)) \geq 0.$$

Now by standard definition

$$\begin{aligned} \cos(\omega(h_i - h_j)) &= \frac{1}{2} \exp\{i\omega(h_i - h_j)\} + \frac{1}{2} \exp\{-i\omega(h_i - h_j)\} \\ &= \frac{1}{2} \exp\{i\omega h_i\} \exp\{-i\omega h_j\} + \frac{1}{2} \exp\{-i\omega h_i\} \exp\{i\omega h_j\} \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{i=1}^d \sum_{j=1}^d h_i h_j \cos(\omega(h_i - h_j)) &= \sum_{i=1}^d \sum_{j=1}^d h_i h_j \left(\frac{1}{2} \exp\{i\omega h_i\} \exp\{-i\omega h_j\} + \frac{1}{2} \exp\{-i\omega h_i\} \exp\{i\omega h_j\} \right) \\ &= \left(\sum_{i=1}^d h_i \exp\{i\omega h_i\} \right) \left(\sum_{j=1}^d h_j \exp\{-i\omega h_j\} \right) \\ &= \left| \sum_{i=1}^d h_i \exp\{i\omega h_i\} \right|^2 \quad \text{as the sums are complex conjugates} \\ &\geq 0 \end{aligned}$$

Of course, we have already established that an even function is non-negative definite if and only if it is the ACVF of a stationary process. Here, we have demonstrated that this is the ACVF of the original harmonic process. Therefore, we can conclude immediately that it is non-negative definite.