

MATH 545: INTRODUCTION TO TIME SERIES ANALYSIS

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CONSTRUCTING STATIONARY AND INVERTIBLE SOLUTIONS

For the stationary ARMA(p, q) model

$$\Phi(B)X_t = \Theta(B)Z_t$$

where $\Phi(z) \neq 0$ for $|z| = 1$, we have

$$X_t = \{\Phi(B)\}^{-1} \Theta(B)Z_t = \Psi(B)Z_t$$

say, so

$$G_X(z) = \sigma_Z^2 \Psi(z) \Psi(z^{-1}) = \sigma_Z^2 \frac{\Theta(z) \Theta(z^{-1})}{\Phi(z) \Phi(z^{-1})}$$

which is well defined and finite as the denominator is non zero. Note that if $\{Z_t\} \sim \text{WN}(0, \sigma_Z^2)$,

$$G_Z(z) = \sigma_Z^2$$

and that this is a unique characteristic of white noise processes. Let

$$\Phi(z) = \prod_{j=1}^p (1 - \xi_j z) \quad \Theta(z) = \prod_{j=1}^q (1 - \omega_j z)$$

We have that the ARMA(p, q) process

- is *causal* if the AR reciprocal roots satisfy $|\xi_j| < 1$ for $j = 1, 2, \dots, p$,
- is *invertible* if the MA reciprocal roots satisfy $|\omega_j| < 1$ for $j = 1, 2, \dots, q$.

and

$$G_X(z) = \sigma_Z^2 \Psi(z) \Psi(z^{-1}) = \sigma_Z^2 \frac{\Theta(z) \Theta(z^{-1})}{\Phi(z) \Phi(z^{-1})}.$$

For arbitrary filters $\Phi(B)$ and $\Theta(B)$, there exist polynomials $\Phi^*(z)$ and $\Theta^*(z)$ and a white noise process $\{Z_t^*\}$ such that

$$\Phi^*(B)X_t = \Theta^*(B)Z_t^*$$

is a causal and invertible representation. Suppose that the reciprocal roots are ordered such that

$$\xi_{r+1}, \dots, \xi_p \quad \omega_{s+1}, \dots, \omega_q$$

lie outside of the unit circle. Let

$$\begin{aligned} \Phi^*(z) &= \Phi(z) \prod_{j=r+1}^p \left(\frac{1 - \xi_j z}{1 - \xi_j^{-1} z} \right)^{-1} = \prod_{j=1}^r (1 - \xi_j z) \prod_{j=r+1}^p (1 - \xi_j^{-1} z) \\ \Theta^*(z) &= \Theta(z) \prod_{j=s+1}^q \left(\frac{1 - \omega_j z}{1 - \omega_j^{-1} z} \right)^{-1} = \prod_{j=1}^s (1 - \omega_j z) \prod_{j=s+1}^q (1 - \omega_j^{-1} z) \end{aligned}$$

so that all the reciprocal roots of these polynomials lie within the unit circle. Consider

$$Z_t^* = \{\Theta^*(B)\}^{-1} \Phi^*(B)X_t.$$

As X_t is stationary, $\{Z_t^*\}$ is also stationary as it is a filtered version of $\{X_t\}$ obtained by filters with reciprocal roots that lie within the unit circle. We have that

$$\begin{aligned}
Z_t^* &= \{\Theta^*(B)\}^{-1} \Phi(B) \left\{ \prod_{j=r+1}^p (1 - \xi_j B)^{-1} (1 - \xi_j^{-1} B) \right\} X_t \\
&= \{\Theta^*(B)\}^{-1} \left\{ \prod_{j=r+1}^p (1 - \xi_j B)^{-1} (1 - \xi_j^{-1} B) \right\} \Phi(B) X_t && \text{rearranging the filters} \\
&= \{\Theta^*(B)\}^{-1} \left\{ \prod_{j=r+1}^p (1 - \xi_j B)^{-1} (1 - \xi_j^{-1} B) \right\} \Theta(B) Z_t && \text{from the model definition} \\
&= \{\Theta^*(B)\}^{-1} \Theta(B) \left\{ \prod_{j=r+1}^p (1 - \xi_j B)^{-1} (1 - \xi_j^{-1} B) \right\} Z_t && \text{rearranging the filters} \\
&= \left\{ \prod_{j=s+1}^q (1 - \omega_j B) (1 - \omega_j^{-1} B)^{-1} \right\} \left\{ \prod_{j=r+1}^p (1 - \xi_j B)^{-1} (1 - \xi_j^{-1} B) \right\} Z_t
\end{aligned}$$

Therefore, $\{Z_t^*\}$ is a filtered version of $\{Z_t\}$. The ACVGF of $\{Z_t^*\}$ is

$$G_{Z^*}(z) = \sigma_Z^2 \Psi^*(z) \Psi^*(z^{-1})$$

where

$$\Psi^*(z) = \left\{ \prod_{j=s+1}^q \frac{1 - \omega_j z}{1 - \omega_j^{-1} z} \right\} \left\{ \prod_{j=r+1}^p \frac{1 - \xi_j^{-1} z}{1 - \xi_j z} \right\}$$

so that

$$\Psi^*(z) \Psi^*(z^{-1}) = \left\{ \prod_{j=s+1}^q \frac{1 - \omega_j z}{1 - \omega_j^{-1} z} \right\} \left\{ \prod_{j=r+1}^p \frac{1 - \xi_j^{-1} z}{1 - \xi_j z} \right\} \left\{ \prod_{j=s+1}^q \frac{1 - \omega_j z^{-1}}{1 - \omega_j^{-1} z^{-1}} \right\} \left\{ \prod_{j=r+1}^p \frac{1 - \xi_j^{-1} z^{-1}}{1 - \xi_j z^{-1}} \right\}.$$

This expression can be simplified by cancelling terms. For example, for $r+1 \leq j \leq p$, consider the terms in ξ_j :

$$\left(\frac{1 - \xi_j^{-1} z}{1 - \xi_j z} \right) \left(\frac{1 - \xi_j^{-1} z^{-1}}{1 - \xi_j z^{-1}} \right) = \xi_j^{-2} \left(\frac{\xi_j - z}{1 - \xi_j z} \right) \left(\frac{z \xi_j - 1}{z - \xi_j} \right) = \xi_j^{-2}.$$

Similar calculations for terms in ω_j leading to ω_j^2 . Hence, noting that complex-valued roots appear in conjugate pairs, we have

$$G_{Z^*}(z) = \sigma_Z^2 \Psi^*(z) \Psi^*(z^{-1}) = \sigma_Z^2 \prod_{j=r+1}^p |\xi_j|^{-2} \prod_{j=s+1}^q |\omega_j|^2 = \sigma_{Z^*}^2$$

say, which is a constant, so $G_{Z^*}(z)$ does not depend on z , and $\{Z_t^*\} \sim \text{WN}(0, \sigma_{Z^*}^2)$. Now, we have from the definition of $\{Z_t^*\}$ that

$$\Phi^*(B) X_t = \Theta^*(B) Z_t^*.$$

$\Phi^*(z)$ is a product of factors where all the reciprocal roots are less than 1 in modulus, and $\Theta^*(z)$ is a product of factors where all the reciprocal roots are less than 1 in modulus. Hence $\{X_t\}$ that solves the original ARMA system

$$\Phi(B) X_t = \Theta(B) Z_t$$

also solves the causal and invertible ARMA system

$$\Phi^*(B)X_t = \Theta^*(B)Z_t^*.$$

defined with respect to the new white noise process $\{Z_t^*\} \sim \text{WN}(0, \sigma_{Z^*}^2)$.

Example: Consider the ARMA(1, 1) system

$$(1 - 4B)X_t = (1 - 3B)Z_t$$

which yields a stationary but non-causal and non-invertible system. Define

$$\Phi^*(z) = (1 - 4z) \frac{(1 - (1/4)z)}{(1 - 4z)} = (1 - (1/4)z) \quad \Theta^*(z) = (1 - 3z) \frac{(1 - (1/3)z)}{(1 - 3z)} = (1 - (1/3)z).$$

The process $\{Z_t^*\}$ is defined by

$$Z_t^* = \{(1 - 3B)(1 - (1/3)B)^{-1}\} \{(1 - 4B)^{-1}(1 - (1/4)B)\} Z_t$$

which we have demonstrated is also white noise with variance

$$\sigma_{Z^*}^2 = \sigma_Z^2 \frac{3^2}{4^2} = \frac{9\sigma_Z^2}{16}$$

The resulting ARMA system is therefore

$$(1 - (1/4)B)X_t = (1 - (1/3)B)Z_t^*$$

which is evidently causal and invertible.

To verify this, note that the system is

$$X_t = 4X_{t-1} + Z_t - 3Z_{t-1}$$

so that

$$X_{t-1} = \frac{1}{4}X_t - \frac{1}{4}Z_t + \frac{3}{4}Z_{t-1}$$

and we can simulate the process backwards in time.

```
n<-100000
sigZ<-2
Z<-rnorm(n,0,sigZ)
X<-rep(0,n)
X[n]<-Z[n]
for(i in (n-1):1){
  X[i]<-0.25*X[i+1]-0.25*Z[i+1]+0.75*Z[i]
}
var(X)

## [1] 2.251164

acf.original<-acf(X,lag.max=25,plot=FALSE)$acf
Xst<-arima.sim(n,model=list(ar=1/4,ma=-1/3),sd=sigZ*(3/4))
var(Xst)

## [1] 2.275756

acf.converted<-acf(Xst,lag.max=25,plot=FALSE)$acf
acf.true<-ARMAacf(ar=c(0.25),ma=c(-1/3),lag.max=25)
round(cbind(acf.true,acf.original,acf.converted),5)
```

##	acf.true	acf.original	acf.converted
## 0	1.00000	1.00000	1.00000
## 1	-0.08088	-0.08140	-0.07941
## 2	-0.02022	-0.02603	-0.01693
## 3	-0.00506	-0.00426	-0.00495
## 4	-0.00126	0.00322	0.00008
## 5	-0.00032	-0.00195	0.00144
## 6	-0.00008	-0.00568	0.00264
## 7	-0.00002	-0.00152	-0.00249
## 8	0.00000	-0.00301	0.00099
## 9	0.00000	0.00228	0.00012
## 10	0.00000	0.00199	-0.00183
## 11	0.00000	0.00824	-0.00242
## 12	0.00000	-0.00021	0.00527
## 13	0.00000	-0.00278	-0.00023
## 14	0.00000	0.00107	0.00267
## 15	0.00000	0.00346	0.00102
## 16	0.00000	0.00115	0.00083
## 17	0.00000	-0.00295	0.00200
## 18	0.00000	0.00091	-0.00272
## 19	0.00000	0.00290	-0.00040
## 20	0.00000	-0.00318	-0.00072
## 21	0.00000	0.00050	0.00197
## 22	0.00000	-0.00135	0.00065
## 23	0.00000	-0.00325	0.00026
## 24	0.00000	-0.00279	-0.00328
## 25	0.00000	0.00156	0.00104