

MATH 545: INTRODUCTION TO TIME SERIES ANALYSIS

WINTER 2025

THE AR(2) PROCESS

Consider the stationary, causal AR(2) process defined in terms of white noise process $\{Z_t\}$ via

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t$$

or equivalently

$$(1 - \xi_1 B)(1 - \xi_2 B)X_t = (1 - (\xi_1 + \xi_2)B + \xi_1 \xi_2 B^2) = Z_t$$

for reciprocal roots ξ_1, ξ_2 , so that

$$\xi_1 + \xi_2 = \phi_1 \quad \xi_1 \xi_2 = -\phi_2. \quad (1)$$

We have by the usual calculation method (multiplying by X_{t-h} , taking expectations) that

$$h = 0 \quad \gamma_X(0) = \phi_1 \gamma_X(1) + \phi_2 \gamma_X(2) + \sigma_Z^2 \quad (2)$$

$$h = 1 \quad \gamma_X(1) = \phi_1 \gamma_X(0) + \phi_2 \gamma_X(1) \quad (3)$$

$$h = 2 \quad \gamma_X(2) = \phi_1 \gamma_X(1) + \phi_2 \gamma_X(0) \quad (4)$$

$$h \geq 3 \quad \gamma_X(h) = \phi_1 \gamma_X(h-1) + \phi_2 \gamma_X(h-2) \quad (5)$$

We solve this system for $\gamma_X(0), \gamma_X(1), \gamma_X(2)$ by direct algebraic manipulation. From (3) and thus from (4) we have

$$\gamma_X(1) = \frac{\phi_1}{1 - \phi_2} \gamma_X(0) \quad \therefore \quad \gamma_X(2) = \frac{\phi_1^2 + (1 - \phi_2)\phi_2}{1 - \phi_2} \gamma_X(0)$$

so from (2)

$$\gamma_X(0) = \frac{(1 - \phi_2)}{1 - \phi_2 - \phi_1^2 - \phi_2(\phi_1^2 + (1 - \phi_2)\phi_2)} \sigma_Z^2.$$

Subsequently we have for $h \geq 3$ the linear recurrence relation given by (5), that is

$$\begin{bmatrix} \gamma_X(h) \\ \gamma_X(h-1) \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \gamma_X(h-1) \\ \gamma_X(h-2) \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} \gamma_X(h-2) \\ \gamma_X(h-3) \end{bmatrix} = \cdots = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix}^{h-2} \begin{bmatrix} \gamma_X(2) \\ \gamma_X(1) \end{bmatrix}$$

To solve the recursion, we diagonalize the matrix using an eigen decomposition, and write

$$\begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}^{-1}.$$

For the eigenvalues, we have the characteristic equation

$$-\lambda(\phi_1 - \lambda) - \phi_2 = 0 \quad \text{that is} \quad \lambda^2 - \lambda\phi_1 - \phi_2 = 0.$$

Hence we have

$$\lambda_1 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} \quad \lambda_2 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

and therefore

$$\lambda_1 + \lambda_2 = \phi_1 \quad \lambda_1 \lambda_2 = -\phi_2.$$

We can therefore conclude by inspecting equation (1) that $(\lambda_1, \lambda_2) \equiv (\xi_1, \xi_2)$.

From the diagonalization, we have that

$$\begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix}^{h-2} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^{h-2} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}^{-1}$$

and hence

$$\begin{aligned} \begin{bmatrix} \gamma_X(h) \\ \gamma_X(h-1) \end{bmatrix} &= \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^{h-2} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}^{-1} \begin{bmatrix} \gamma_X(2) \\ \gamma_X(1) \end{bmatrix} \\ &= \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} \lambda_1^{h-2} & 0 \\ 0 & \lambda_2^{h-2} \end{bmatrix} \frac{1}{v_{11}v_{22} - v_{12}v_{21}} \begin{bmatrix} v_{22}\gamma_X(2) - v_{12}\gamma_X(1) \\ v_{11}\gamma_X(1) - v_{21}\gamma_X(2) \end{bmatrix} \\ &= \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \frac{1}{v_{11}v_{22} - v_{12}v_{21}} \begin{bmatrix} (v_{22}\gamma_X(2) - v_{12}\gamma_X(1))\lambda_1^{h-2} \\ (v_{11}\gamma_X(1) - v_{21}\gamma_X(2))\lambda_2^{h-2} \end{bmatrix} \end{aligned}$$

so therefore for $h \geq 3$

$$\gamma_X(h) = \frac{1}{v_{11}v_{22} - v_{12}v_{21}} \left[v_{11}(v_{22}\gamma_X(2) - v_{12}\gamma_X(1))\lambda_1^{h-2} + v_{12}(v_{11}\gamma_X(1) - v_{21}\gamma_X(2))\lambda_2^{h-2} \right] = c_1\xi_1^{h-2} + c_2\xi_2^{h-2}$$

say, as $\xi_1 = \lambda_1$ and $\xi_2 = \lambda_2$, where

$$c_1 = \frac{v_{11}(v_{22}\gamma_X(2) - v_{12}\gamma_X(1))}{v_{11}v_{22} - v_{12}v_{21}} \quad c_2 = \frac{v_{12}(v_{11}\gamma_X(1) - v_{21}\gamma_X(2))}{v_{11}v_{22} - v_{12}v_{21}}.$$

To compute $v_{11}, v_{12}, v_{21}, v_{22}$ we need the eigenvectors; by a standard result

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ has eigenvectors } \begin{bmatrix} b \\ \lambda_1 - a \end{bmatrix} \text{ and } \begin{bmatrix} \lambda_2 - d \\ c \end{bmatrix}$$

so in our case these are proportional to

$$\begin{bmatrix} \phi_2 \\ \lambda_1 - \phi_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}.$$

We then normalize these vectors to give

$$v_{11} = \frac{\phi_2}{\sqrt{\phi_2^2 + (\lambda_1 - \phi_1)^2}} \quad v_{21} = \frac{(\lambda_1 - \phi_1)}{\sqrt{\phi_2^2 + (\lambda_1 - \phi_1)^2}} \quad v_{12} = \frac{\lambda_2}{\sqrt{\lambda_2^2 + 1}} \quad v_{22} = \frac{1}{\sqrt{\lambda_2^2 + 1}}$$

We therefore have an explicit formula for $\gamma_X(h)$, $h = 0, 1, 2, \dots$

Note: if $\xi_1 = \xi_2 = \alpha$, and we have repeated reciprocal roots, then the eigenvectors are identical. In this case, the general solution to (5) for $h \geq 3$ takes the form

$$\begin{aligned} \gamma_X(h) &= (c_1 + c_2(h-2))\alpha^{h-2} \\ \implies \gamma_X(3) &= (c_1 + c_2)\alpha \equiv \phi_1\gamma_X(2) + \phi_2\gamma_X(1) \\ \gamma_X(4) &= (c_1 + 2c_2)\alpha^2 \equiv \phi_1\gamma_X(3) + \phi_2\gamma_X(2) = (\phi_1^2 + \phi_2)\gamma_X(2) + \phi_1\phi_2\gamma_X(1) \end{aligned}$$

which are two equations in the two unknown constants which can be used to identify c_1 and c_2 , that is

$$\begin{bmatrix} \alpha & \alpha \\ \alpha^2 & 2\alpha^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \phi_1\gamma_X(2) + \phi_2\gamma_X(1) \\ (\phi_1^2 + \phi_2)\gamma_X(2) + \phi_1\phi_2\gamma_X(1) \end{bmatrix}.$$

EXAMPLE: Consider the AR(2) process

$$X_t = 1.2X_{t-1} - 0.35X_{t-2} + Z_t.$$

Then

$$\Phi(z) = (1 - 1.2z + 0.35z^2) = (1 - 0.7z)(1 - 0.5z)$$

so $\xi_1 = 0.7, \xi_2 = 0.5$.

```

phi1<-1.2;phi2<--0.35
gam0<-(1-phi2)/(1-phi2-phi1^2-phi2*(phi1^2+(1-phi2)*phi2))
gam1<-phi1/(1-phi2)*gam0
gam2<-((phi1^2+(1-phi2)*phi2)/(1-phi2))*gam0

A<-matrix(c(phi1,1,phi2,0),2,2);eigen(A)

+ eigen() decomposition
+ $values
+ [1] 0.7 0.5
+
+ $vectors
+      [,1]      [,2]
+ [1,] 0.5734623 0.4472136
+ [2,] 0.8192319 0.8944272

al1<-eigen(A)$values[1];al2<-eigen(A)$values[2]
v1<-c(A[1,2],al1-A[1,1]);v1<-v1/sqrt(sum(v1^2))
v2<-c(al2-A[2,2],A[2,1]);v2<-v2/sqrt(sum(v2^2))
cbind(v1,v2)

+      v1      v2
+ [1,] -0.5734623 0.4472136
+ [2,] -0.8192319 0.8944272

v11<-v1[1];v21<-v1[2]
v12<-v2[1];v22<-v2[2]

P<-matrix(c(v11,v21,v12,v22),2,2)
Pinv<-solve(P)
P %%% diag(c(al1,al2)) %%% Pinv

+      [,1]      [,2]
+ [1,] 1.2 -0.34999999999999995337063
+ [2,] 1.0 0.00000000000000004440892

c1<-v11*(v22*gam2-v12*gam1)/(v11*v22-v12*v21)
c2<-v12*(v11*gam1-v21*gam2)/(v11*v22-v12*v21)

gammaX<-rep(0,11)
gammaX[(3+1):(10+1)]<-c1*al1^(c(3:10)-2)+c2*al2^(c(3:10)-2)
gammaX[1:3]<-c(gam0,gam1,gam2)

rhoX<-gammaX/gammaX[1];rhoX

+ [1] 1.00000000 0.88888889 0.71666667 0.54888889 0.40783333 0.29728889
+ [7] 0.21400500 0.15275489 0.10840412 0.07662073 0.05400343

true.rhoX<-ARMAacf(ar=c(phi1,phi2),lag.max=10);as.numeric(true.rhoX)

+ [1] 1.00000000 0.88888889 0.71666667 0.54888889 0.40783333 0.29728889
+ [7] 0.21400500 0.15275489 0.10840412 0.07662073 0.05400343

```

EXAMPLE: Consider the AR(2) process

$$X_t = X_{t-1} - 0.25X_{t-2} + Z_t.$$

Then

$$\Phi(z) = (1 - z + 0.25z^2) = (1 - 0.5z)(1 - 0.5z)$$

so $\xi_1 = \xi_2 = 0.5$.

```
phi1<-1.0;phi2<--0.25
gam0<-(1-phi2)/(1-phi2-phi1^2-phi2*(phi1^2+(1-phi2)*phi2))
gam1<-phi1/(1-phi2)*gam0
gam2<-((phi1^2+(1-phi2)*phi2)/(1-phi2))*gam0

A<-matrix(c(phi1,1,phi2,0),2,2);eigen(A)

+ eigen() decomposition
+ $values
+ [1] 0.5 0.5
+
+ $vectors
+      [,1]      [,2]
+ [1,] 0.4472136 0.4472136
+ [2,] 0.8944272 0.8944272

al1<-eigen(A)$values[1];al2<-eigen(A)$values[2]
v1<-c(A[1,2],al1-A[1,1]);v1<-v1/sqrt(sum(v1^2))
v2<-c(al2-A[2,2],A[2,1]);v2<-v2/sqrt(sum(v2^2))
cbind(v1,v2)

+      v1      v2
+ [1,] -0.4472136 0.4472136
+ [2,] -0.8944272 0.8944272

A<-matrix(c(al1,al1^2,al1,2*al1^2),2,2)
bvec<-c(phi1*gam2+phi2*gam1,(phi1^2+phi2)*gam2 + phi1*phi2*gam1)
cvec<-solve(A,bvec)
cvec

+ [1] 1.6296296 0.4444444

c1<-cvec[1]
c2<-cvec[2]

gammaX<-rep(0,11)
gammaX[1:3]<-c(gam0,gam1,gam2)
gammaX[(3+1):(10+1)]<-(c1+c2*(c(3:10)-2))*al1^(c(3:10)-2)

rhoX<-gammaX/gammaX[1];rhoX

+ [1] 1.000000000 0.800000000 0.550000000 0.350000000 0.212500000 0.125000000
+ [7] 0.071875000 0.040625000 0.022656250 0.012500000 0.006835937

true.rhoX<-ARMAacf(ar=c(phi1,phi2),lag.max=10);as.numeric(true.rhoX)

+ [1] 1.000000000 0.800000000 0.550000000 0.350000000 0.212500000 0.125000000
+ [7] 0.071875000 0.040625000 0.022656250 0.012500000 0.006835937
```