

MATH 545: INTRODUCTION TO TIME SERIES ANALYSIS

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THE AR(1) PROCESS

The AR(1) process $\{X_t\}$ is defined via the model equation

$$X_t = \phi X_{t-1} + Z_t \quad t \in \mathbb{Z} \quad (1)$$

where $\{Z_t\}$ is a white noise process, $\{Z_t\} \sim \text{WN}(0, \sigma_Z^2)$ say. That is, $\{X_t\}$ is the solution to (1). We can construct the form of the solution by applying the recursive approach. Assume first that $|\phi| < 1$.

$$\begin{aligned} X_t &= \phi X_{t-1} + Z_t \\ &= \phi^2 X_{t-2} + (Z_t + \phi Z_{t-1}) \\ &= \phi^3 X_{t-3} + (Z_t + \phi Z_{t-1} + \phi^2 Z_{t-2}) \\ &\vdots \\ &= \phi^{k+1} X_{t-k-1} + \sum_{j=0}^k \phi^j Z_{t-j} \end{aligned}$$

as $|\phi| < 1$. We can verify that this limit exists, as differences in the partial sums converge to zero, where convergence is in the mean-square metric. That is, for $l > k$

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{j=0}^l \phi^j Z_{t-j} - \sum_{j=0}^k \phi^j Z_{t-j} \right)^2 \right] &= \mathbb{E} \left[\left(\sum_{j=k+1}^l \phi^j Z_{t-j} \right)^2 \right] \\ &= \mathbb{E} \left[\sum_{j=k+1}^l \phi^{2j} Z_{t-j}^2 + \sum_{j=k+1}^l \sum_{i \neq j} \phi^{j+k} Z_{t-j} Z_{t-i} \right] \\ &= \sum_{j=k+1}^l \phi^{2j} \sigma_Z^2 \quad \text{as } Z_t \sim \text{WN}(0, \sigma_Z^2). \\ &\leq \phi^{2(k+1)} \sigma_Z^2 \sum_{j=0}^{\infty} \phi^{2j} \quad \text{all terms positive} \\ &\longrightarrow 0 \end{aligned}$$

as $k \longrightarrow \infty$ as $|\phi| < 1$. Thus, with limit taken in the *mean-square* sense, we may write for all t

$$X_t = \lim_{k \longrightarrow \infty} \sum_{j=0}^k \phi^j Z_{t-j} = \sum_{j=0}^{\infty} \phi^j Z_{t-j}.$$

Here we also have that the limit exists in the *almost sure* sense; we have that

$$\sum_{j=0}^{\infty} |\phi^j| = \sum_{j=0}^{\infty} |\phi|^j = \frac{1}{1 - |\phi|} < \infty$$

and $\mathbb{E}[|Z_t|] < \infty$ for all t , and so the conditions of the general theory hold.

This solution is unique up to mean square; if $\{Y_t\}$ is another stationary solution, we know by definition that $\{Y_t\}$ satisfies the recursion. That is, we have using the above method that $\{Y_t\}$ satisfies

$$Y_t = \phi^{k+1}Y_{t-k-1} + \sum_{j=0}^k \phi^j Z_{t-j}$$

so therefore

$$Y_t - \sum_{j=0}^k \phi^j Z_{t-j} = \phi^{k+1}Y_{t-k-1}.$$

Squaring and taking expectations, we have

$$\mathbb{E} \left[\left(Y_t - \sum_{j=0}^k \phi^j Z_{t-j} \right)^2 \right] = \phi^{2(k+1)} \mathbb{E}[Y_{t-k-1}^2]$$

Now $\mathbb{E}[Y_{t-k}^2] < \infty$ by stationarity, so taking the limit as $k \rightarrow \infty$, as $|\phi| < 1$, we have

$$\mathbb{E} \left[\left(Y_t - \sum_{j=0}^k \phi^j Z_{t-j} \right)^2 \right] \rightarrow 0.$$

Thus Y_t is also equal to the mean-square limit. This also establishes that $Y_t \stackrel{\text{a.s.}}{=} X_t$ as

$$\mathbb{E}[(X_t - Y_t)^2] = 0$$

as $X_t - Y_t$ has mean zero by stationarity, and the fact that we know Y_t satisfies the recursion.