

PRACTICE MID TERM: SOLUTIONS

1. If

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} \quad t \in \mathbb{Z}$$

with the constraint on the ψ_j values, then the process is stationary and has mean zero by direct calculation. Then for $h \geq 0$

$$\begin{aligned} \gamma_X(h) &= \mathbb{E}[X_t X_{t+h}] = \mathbb{E} \left[\left(\sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} \right) \left(\sum_{k=-\infty}^{\infty} \psi_k Z_{t+h-k} \right) \right] \\ &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \mathbb{E}[Z_{t-j} Z_{t+h-k}] \\ &= \sigma_Z^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \end{aligned}$$

by linearity of expectations, and as

$$\mathbb{E}[Z_{t-j} Z_{t+h-k}] = \begin{cases} \sigma_Z^2 & k = h + j \\ 0 & \text{otherwise} \end{cases}$$

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If $X_t = Z_{t+1} + Z_t + Z_{t-1}$, then we have the linear process representation with $\psi_{-1} = \psi_0 = \psi_1 = 1$, with all other ψ_j zero. Hence

$$\begin{aligned} h = 0 \quad \gamma_X(0) &= \sigma_Z^2 \sum_{j=-\infty}^{\infty} \psi_j^2 = \sigma_Z^2 (\psi_0^2 + \psi_1^2 + \psi_2^2) = 3\sigma_Z^2 \\ h = 1 \quad \gamma_X(1) &= \sigma_Z^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+1} = \sigma_Z^2 (\psi_{-1}\psi_0 + \psi_0\psi_1) = 2\sigma_Z^2 \\ h = 2 \quad \gamma_X(2) &= \sigma_Z^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+2} = \sigma_Z^2 (\psi_{-1}\psi_1) = \sigma_Z^2 \end{aligned}$$

with $\gamma_X(h) = 0$ for $h > 2$; note also that for $h < 0$, $\gamma_X(h) = \gamma_X(-h)$.

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2. (a) We have the AR(2) process

$$X_t - 1.2X_{t-1} + 0.35X_{t-2} = (1 - 1.2B + 0.35B^2)X_t = (1 - 0.7B)(1 - 0.5B)X_t = Z_t.$$

Thus the reciprocal roots are $\xi_1 = 0.7$ and $\xi_2 = 0.5$, which are both less than one in modulus. This is a necessary (and sufficient) condition for the process to be stationary and causal.

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(b) We have the AR(3) process

$$X_t - 3X_{t-1} + 3X_{t-2} - X_{t-3} = (1 - 3B + 3B^2 - B^3)X_t = (1 - B)^3 X_t = Z_t.$$

Thus the reciprocal roots are $\xi_1 = \xi_2 = \xi_3 = 1$ thus the process is not stationary. No mean-square convergent linear process representation of $\{X_t\}$ exists, so the process is technically not causal, however X_t is defined only in terms of previous values of X .

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3. Rearranging, we have

$$\left(1 - \frac{1}{2}B\right) X_t = \left(1 + \frac{1}{2}B\right) Z_t$$

so by the usual convergent geometric expansion, we have

$$X_t = \left(1 - \frac{1}{2}B\right)^{-1} \left(1 + \frac{1}{2}B\right) Z_t = \left(1 + \frac{1}{2}B + \left(\frac{1}{2}\right)^2 B^2 + \dots\right) \left(1 + \frac{1}{2}B\right) Z_t = \Psi(B) Z_t$$

say. Thus we identify the linear process representation by inspecting coefficients.

$$\begin{aligned} 1 \quad \psi_0 &= 1 \\ B \quad \psi_1 &= \frac{1}{2} + \frac{1}{2} = 1 \\ B^2 \quad \psi_2 &= \left(\frac{1}{2}\right)^2 + \frac{1}{2} \frac{1}{2} = \frac{1}{2} \\ &\vdots \\ B^j \quad \psi_j &= \left(\frac{1}{2}\right)^j + \frac{1}{2} \left(\frac{1}{2}\right)^{j-1} = \left(\frac{1}{2}\right)^{j-1} \\ &\vdots \end{aligned}$$

Thus we have

$$\begin{aligned} \gamma_X(0) &= \sigma_Z^2 \sum_{j=0}^{\infty} \psi_j^2 = \sigma_Z^2 \left(1 + \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^{2(j-1)}\right) = \sigma_Z^2 \left(1 + \frac{1}{1 - (1/2)^2}\right) = \frac{7}{3} \sigma_Z^2 \\ \gamma_X(h) &= \sigma_Z^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} = \sigma_Z^2 \left(\left(\frac{1}{2}\right)^{h-1} + \left(\frac{1}{2}\right)^h \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^{2(j-1)}\right) = \frac{5}{3} \sigma_Z^2 \left(\frac{1}{2}\right)^{h-1} \quad h = 1, 2, \dots \end{aligned}$$

so therefore

$$\rho_X(h) = \frac{5}{7} \left(\frac{1}{2}\right)^{h-1} \quad h = 1, 2, \dots$$

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For the second process, we have

$$\left(1 - \frac{1}{2}B\right) X_t = \left(1 - \frac{1}{2}B\right) Z_t$$

and therefore having the inverse operator $\left(1 - \frac{1}{2}B\right)^{-1}$ operate on left and right-hand sides, we see that $\{X_t\}$ is mean-square identical to $\{Z_t\}$, and hence is white noise.

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4. We verify stationarity from first principles: we have

$$\mathbb{E}[X_t] = \mathbb{E}[U \cos(\omega t) + V \sin(\omega t)] = \mathbb{E}[U] \cos(\omega t) + \mathbb{E}[V] \sin(\omega t) = 0 \quad \forall t$$

and

$$\text{Var}[X_t] = \text{Var}[U \cos(\omega t) + V \sin(\omega t)] = \text{Var}[U] \cos^2(\omega t) + \text{Var}[V] \sin^2(\omega t) = 1 \quad \forall t.$$

Finally, for $h = 1, 2, \dots$

$$\begin{aligned}\gamma_X(h) &= \mathbb{E}[X_t X_{t+h}] = \mathbb{E}[(U \cos(\omega t) + V \sin(\omega t))(U \cos(\omega(t+h)) + V \sin(\omega(t+h)))] \\ &= \cos(\omega t) \cos(\omega(t+h)) + \sin(\omega t) \sin(\omega(t+h))\end{aligned}$$

as $\mathbb{E}[U^2] = \mathbb{E}[V^2] = 1$ and $\mathbb{E}[UV] = 0$. But using the double angle formulae

$$\cos(\omega(t+h)) = \cos(\omega t) \cos(\omega h) - \sin(\omega t) \sin(\omega h) \quad \sin(\omega(t+h)) = \cos(\omega t) \sin(\omega h) + \sin(\omega t) \cos(\omega h)$$

and thus, collecting terms together, we have that

$$\gamma_X(h) = \cos(\omega h).$$

Thus $\{X_t\}$ is stationary.

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