

MATH 545 - MID-TERM EXAMINATION: SOLUTIONS

1. We have that for all t ,

$$\mathbb{E}[X_t] = \mathbb{E}[U \cos(\omega t)] + \mathbb{E}[V \sin(\omega t)] + \mathbb{E}[Z_t] = 0$$

and by direct calculation

$$\begin{aligned} \text{Cov}[X_t, X_{t+h}] &= \mathbb{E}[X_t X_{t+h}] \\ &= \mathbb{E}[(U \cos(\omega t) + V \sin(\omega t) + Z_t)(U \cos(\omega(t+h)) + V \sin(\omega(t+h)) + Z_{t+h})] \\ &= \mathbb{E}[U^2] \cos(\omega t) \cos(\omega(t+h)) + \mathbb{E}[V^2] \sin(\omega t) \sin(\omega(t+h)) + \mathbb{E}[Z_t Z_{t+h}] \\ &= \cos(\omega t) \cos(\omega(t+h)) + \sin(\omega t) \sin(\omega(t+h)) + \mathbb{E}[Z_t Z_{t+h}] \\ &= \begin{cases} \cos(\omega(-h)) + \sigma_Z^2 & h = 0 \\ \cos(\omega(-h)) & h \neq 0 \end{cases} \end{aligned}$$

where the third line follows from the second as for all t ,

$$\mathbb{E}[UV] = \mathbb{E}[UZ_t] = \mathbb{E}[VZ_t] = 0$$

by uncorrelatedness. Hence $\{X_t\}$ is stationary.

10 MARKS

2. (a) We have that

$$X_t = X_{t-1} - 0.24X_{t-2} + Z_t.$$

that is, the AR polynomial is

$$\Phi(z) = 1 - z + 0.24z^2 = (1 - 0.4z)(1 - 0.6z)$$

and the reciprocal roots are 0.4 and 0.6, which are both less than 1 in absolute value. Thus $\{X_t\}$ is stationary and causal.

5 MARKS

(b) Here

$$X_t = 2X_{t-1} - X_{t-2} + Z_t.$$

that is, the AR polynomial is

$$\Phi(z) = 1 - 2z + z^2 = (1 - z)^2$$

so the reciprocal roots are both equal to 1, and the process is not stationary.

5 MARKS

3. We can write the process as

$$(1 - (1/2)B)X_t = (1 + B)Z_t$$

so by inspection of the reciprocal roots, the process is causal but not invertible.

4 MARKS

We have that the generating function of coefficients $\Psi(z)$ must satisfy

$$\Psi(z) = \sum_{j=0}^{\infty} \psi_j z^j = (1 - (1/2)z)^{-1}(1 + z) = \left(1 + \frac{1}{2}z + \frac{1}{4}z^2 + \cdots + \frac{1}{2^j}z^j + \cdots\right)(1 + z)$$

so that $\psi_0 = 1$

$$j = 1 \quad \psi_1 = (1 + 1/2) = 3/2$$

$$j \geq 2 \quad \psi_j = (1/2^j + 1/2^{j-1}) = (3/2)(1/2)^{j-1}$$

6 MARKS

4. Here the MA(1) process is clearly zero mean, and

$$\begin{aligned}\mathbb{E}[X_t X_{t+h}] &= \mathbb{E}[(Z_t - Z_{t-1})(Z_{t+h} - Z_{t+h-1})] \\ &= \mathbb{E}[Z_t Z_{t+h}] - \mathbb{E}[Z_t Z_{t+h-1}] - \mathbb{E}[Z_{t-1} Z_{t+h}] + \mathbb{E}[Z_{t-1} Z_{t+h-1}] \\ &= \begin{cases} 2\sigma_Z^2 & h = 0 \\ -\sigma_Z^2 & h = \pm 1 \\ 0 & |h| > 1 \end{cases}\end{aligned}$$

Hence

$$G_X(z) = \sigma_Z^2(-1/z + 2 - z)$$

for all $z \in \mathbb{C}$.

6 MARKS

We have that $\{X_t\}$ is zero mean, and

$$\begin{aligned}\mathbb{E}[X_t X_{t+h}] &= \mathbb{E}[(X_{1,t} + X_{2,t})(X_{1,t+h} + X_{2,t+h})] \\ &= \mathbb{E}[X_{1,t} X_{1,t+h}] + \mathbb{E}[X_{2,t} X_{2,t+h}] && \text{by independence} \\ &= \gamma_{X_1}(h) + \gamma_{X_2}(h)\end{aligned}$$

say. Therefore we have that

$$\gamma_X(h) = \begin{cases} \sigma_{Z_1}^2(1 + \theta_{1,1}^2) + \sigma_{Z_2}^2(1 + \theta_{2,1}^2) & h = 0 \\ \sigma_{Z_1}^2 \theta_{1,1} + \sigma_{Z_2}^2 \theta_{2,1} & h = \pm 1 \\ 0 & |h| > 1 \end{cases}$$

Therefore $\{X_t\}$ has a representation as an MA(1) process. The parameter of this process is defined by the ACF at lag 1.

$$\frac{\theta_1}{1 + \theta_1^2} = \frac{\sigma_{Z_1}^2 \theta_{1,1} + \sigma_{Z_2}^2 \theta_{2,1}}{\sigma_{Z_1}^2(1 + \theta_{1,1}^2) + \sigma_{Z_2}^2(1 + \theta_{2,1}^2)} = \rho_1$$

say, so that

$$\rho_1 \theta_1^2 - \theta_1 + \rho_1 = 0$$

that is

$$\theta_1 = \frac{1 \pm \sqrt{1 - 4\rho_1^2}}{2}$$

The driving white noise variance is

$$\sigma_Z^2 = \frac{\sigma_{Z_1}^2 \theta_{1,1} + \sigma_{Z_2}^2 \theta_{2,1}}{\theta_1}.$$

4 MARKS