

MATH 545 - MID-TERM EXAMINATION

All questions carry 10 marks.

The total mark available is 40; rescaling of the final mark may be carried out.

In the following questions, $\{Z_t\} \sim \text{WN}(0, \sigma_Z^2)$, with $t \in \mathbb{Z}$; ACVF refers to the *autocovariance* function. Here questions concerning stationarity refer to *weak* stationarity.

1. State, with justification, whether the following time series processes are stationary and causal:

- (a) $\{X_t\}$ defined implicitly for all t by the equation 5 MARKS

$$X_t = X_{t-1} - X_{t-2} + Z_t.$$

- (b) $\{X_t\}$ defined implicitly for all t by the equation 5 MARKS

$$X_t = 0.4X_{t-1} - 0.03X_{t-2} + Z_t - Z_{t-1}.$$

2. Consider the stationary and causal ARMA(1, 2) process

$$X_t = \frac{1}{3}X_{t-1} + Z_t + 2Z_{t-1} + Z_{t-2}.$$

Derive the equations that define a linear system that allows calculation of the ACVF

$$\gamma_X(h) = \mathbb{E}[X_t X_{t-h}]$$

for $h = 0, 1, 2$.

8 MARKS

Explain how to compute $\gamma_X(h)$ for $h > 2$.

2 MARKS

3. Consider the AR(1) process

$$X_t = 0.9X_{t-1} + Z_t.$$

Derive the form of the optimal linear h -step-ahead forecast, $\hat{X}_{n+h}$, derived under the minimum MSE criterion, based on variables $X_1, \dots, X_n$. 6 MARKS

Find the minimum value of the MSE

$$\mathbb{E}[(X_{n+h} - \hat{X}_{n+h})^2]$$

and comment on the behaviour of this quantity as $h \rightarrow \infty$.

4 MARKS

4. The time series process $\{X_t\}$ defined for all $t \in \mathbb{Z}$ by

$$X_t = U \cos(2t) + V \sin(2t) + Z_t + Z_{t-1}$$

where U and V are uncorrelated random variables with expectation zero and variance one, and where U and V are uncorrelated with $\{Z_t\}$. Find, for each t , $\mathbb{E}[X_t X_{t+h}]$ for $h \geq 0$. Hence state whether $\{X_t\}$ is stationary. 10 MARKS

Hint: recall that

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$