

MATH 545 - MID-TERM EXAMINATION

All questions carry 10 marks.

The total mark available is 40; rescaling of the final mark may be carried out.

In the following questions, $\{Z_t\} \sim \text{WN}(0, \sigma_Z^2)$, with $t \in \mathbb{Z}$; ACVF refers to the *autocovariance* function. Here questions concerning stationarity refer to *weak* stationarity.

1. State, with justification, whether the following time series processes are stationary and causal:

(a) $\{X_t\}$ defined implicitly for all t by the equation

5 MARKS

$$X_t = X_{t-1} - X_{t-2} + Z_t.$$

Solution: Here the AR polynomial is

$$(1 - B + B^2)X_t = Z_t$$

and hence the roots of the equivalent $\Phi(z) = 0$ equation are

$$\frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}.$$

The modulus of each root is therefore 1, and the model is nonstationary. Note also that we know from the study of quadratic polynomials that here the product of the roots is 1 (the coefficient of the constant term), and therefore can deduce that the model is nonstationary directly.

(b) $\{X_t\}$ defined implicitly for all t by the equation

5 MARKS

$$X_t = 0.4X_{t-1} - 0.03X_{t-2} + Z_t - Z_{t-1}.$$

Solution: Here the AR polynomial is

$$\Phi(z) = (1 - 0.4z + 0.03z^2) = (1 - 0.1z)(1 - 0.3z)$$

and hence the reciprocal roots are 0.1 and 0.3, which are both less than one in modulus, and the process is stationary. We do not need to consider the MA polynomial when considering stationarity.

2. Consider the stationary and causal ARMA(1, 2) process

$$X_t = \frac{1}{3}X_{t-1} + Z_t + 2Z_{t-1} + Z_{t-2}.$$

Derive the equations that define a linear system that allows calculation of the ACVF

$$\gamma_X(h) = \mathbb{E}[X_t X_{t-h}]$$

for $h = 0, 1, 2$.

8 MARKS

Explain how to compute $\gamma_X(h)$ for $h > 2$.

2 MARKS

Solution: We compute using the technique of multiplying through and taking expectations: for $h \geq 0$

$$\mathbb{E}[X_t X_{t-h}] = \frac{1}{3} \mathbb{E}[X_{t-1} X_{t-h}] + \mathbb{E}[Z_t X_{t-h}] + 2\mathbb{E}[Z_{t-1} X_{t-h}] + \mathbb{E}[Z_{t-2} X_{t-h}].$$

and as $\mathbb{E}[Z_t X_{t-h}] = \mathbb{E}[Z_{t-1} X_{t-h}] = \mathbb{E}[Z_{t-2} X_{t-h}] = 0$ if $h \geq 3$, we have for those values that

$$\gamma_X(h) = \frac{1}{3}\gamma_X(h-1)$$

For $h = 0, 1, 2$ we have

$$\begin{aligned}\mathbb{E}[X_t^2] &= \frac{1}{3}\mathbb{E}[X_{t-1}X_t] + \mathbb{E}[Z_t X_t] + 2\mathbb{E}[Z_{t-1}X_t] + \mathbb{E}[Z_{t-2}X_t] \\ \mathbb{E}[X_t X_{t-1}] &= \frac{1}{3}\mathbb{E}[X_{t-1}X_{t-1}] + \mathbb{E}[Z_t X_{t-1}] + 2\mathbb{E}[Z_{t-1}X_{t-1}] + \mathbb{E}[Z_{t-2}X_{t-1}] \\ \mathbb{E}[X_t X_{t-2}] &= \frac{1}{3}\mathbb{E}[X_{t-1}X_{t-2}] + \mathbb{E}[Z_t X_{t-2}] + 2\mathbb{E}[Z_{t-1}X_{t-2}] + \mathbb{E}[Z_{t-2}X_{t-2}]\end{aligned}$$

and as $\mathbb{E}[Z_s X_t] = 0$ unless $t > s$ in which case $\mathbb{E}[Z_s X_t] = \sigma_Z^2$, we have

$$\begin{aligned}\gamma_X(0) &= \frac{1}{3}\gamma_X(1) + (1 + 2\psi_1 + \psi_2)\sigma_Z^2 \\ \gamma_X(1) &= \frac{1}{3}\gamma_X(0) + (2 + \psi_1)\sigma_Z^2 \\ \gamma_X(2) &= \frac{1}{3}\gamma_X(1) + \sigma_Z^2\end{aligned}$$

where ψ_1 and ψ_2 are the $j = 1$ and $j = 2$ linear process coefficients in the expansion of $\{X_t\}$. This linear system can be solved numerically.

For $h > 2$, we use the recursion

$$\gamma_X(h) = \frac{1}{3}\gamma_X(h-1)$$

directly.

3. Consider the AR(1) process

$$X_t = 0.9X_{t-1} + Z_t.$$

Derive the form of the optimal linear h -step-ahead forecast, $\hat{X}_{n+h}$, derived under the minimum MSE criterion, based on variables $X_1, \dots, X_n$. 6 MARKS

Find the minimum value of the MSE

$$\mathbb{E}[(X_{n+h} - \hat{X}_{n+h})^2]$$

and comment on the behaviour of this quantity as $h \rightarrow \infty$. 4 MARKS

Solution: In this case, we may deduce immediately that as

$$\hat{X}_{n+h} = \mathbb{E}[X_{n+h} | X_{1:n}]$$

we must have by iterated expectation that

$$\begin{aligned}\mathbb{E}[X_{n+h} | X_{1:n}] &= \mathbb{E}_{X_{n+h-1} | X_{1:n}}[\mathbb{E}[X_{n+h} | X_{n-h+1}, X_{1:n}] | X_{1:n}] \\ &= \mathbb{E}_{X_{n+h-1} | X_{1:n}}[\phi X_{n+h-1} | X_{1:n}] \\ &= \phi \mathbb{E}[X_{n+h-1} | X_{1:n}]\end{aligned}$$

so by recursion

$$\hat{X}_{n+h} = \phi^h \mathbb{E}[X_n | X_{1:n}] = \phi^h X_n$$

We then have that

$$\begin{aligned} \mathbb{E}[(X_{n+h} - \hat{X}_{n+h})^2] &= \mathbb{E}[(X_{n+h} - \phi^h X_n)^2] \\ &= \mathbb{E}[X_{n+h}^2] - 2\phi^h \mathbb{E}[X_n X_{n+h}] + \phi^{2h} \mathbb{E}[X_n^2] \\ &= \gamma_X(0) - 2\phi^h \gamma_X(h) + \phi^{2h} \gamma_X(0) \end{aligned}$$

But $\gamma_X(h) = \phi^h \gamma_X(0)$, so this becomes

$$\mathbb{E}[(X_{n+h} - \hat{X}_{n+h})^2] = \gamma_X(0)(1 - \phi^h) \longrightarrow \gamma_X(0)$$

as $h \longrightarrow \infty$.

4. The time series process $\{X_t\}$ defined for all $t \in \mathbb{Z}$ by

$$X_t = U \cos(2t) + V \sin(2t) + Z_t + Z_{t-1}$$

where U and V are uncorrelated random variables with expectation zero and variance one, and where U and V are uncorrelated with $\{Z_t\}$. Find, for each t , $\mathbb{E}[X_t X_{t+h}]$ for $h \geq 0$. Hence state whether $\{X_t\}$ is stationary. 10 MARKS

Hint: recall that

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

Solution: We have that for all t ,

$$\mathbb{E}[X_t] = \mathbb{E}[U \cos(\omega t)] + \mathbb{E}[V \sin(\omega t)] + \mathbb{E}[Z_t + Z_{t-1}] = 0$$

and by direct calculation

$$\begin{aligned} \text{Cov}[X_t, X_{t+h}] &= \mathbb{E}[X_t X_{t+h}] \\ &= \mathbb{E}[(U \cos(2t) + V \sin(2t) + Z_t + Z_{t-1})(U \cos(2(t+h)) + V \sin(2(t+h)) + Z_{t+h} + Z_{t+h-1})] \\ &= \mathbb{E}[U^2] \cos(2t) \cos(2(t+h)) + \mathbb{E}[V^2] \sin(2t) \sin(2(t+h)) + \mathbb{E}[(Z_t + Z_{t-1})(Z_{t+h} + Z_{t+h-1})] \\ &= \cos(2t) \cos(2(t+h)) + \sin(2t) \sin(2(t+h)) + \mathbb{E}[(Z_t + Z_{t-1})(Z_{t+h} + Z_{t+h-1})] \\ &= \begin{cases} \cos(2(-h)) + 2\sigma_Z^2 & h = 0 \\ \cos(2(-h)) + \sigma_Z^2 & h = \pm 1 \\ \cos(2(-h)) & |h| \geq 2 \end{cases} \end{aligned}$$

where the third line follows from the second as for all t ,

$$\mathbb{E}[UV] = \mathbb{E}[UZ_t] = \mathbb{E}[VZ_t] = 0$$

by uncorrelatedness. Hence $\{X_t\}$ is stationary.