

MATH 545: INTRODUCTION TO TIME SERIES ANALYSIS  
WINTER 2025

EXERCISES 4

*Not for assessment.*

1. As quoted in lectures, an absolutely summable function  $\gamma(h)$  is the ACVF for a stationary process if and only if it is even and

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h) \geq 0$$

for all  $\lambda$  in the interval  $(-\pi, \pi]$ . Use this result to assess whether the function

$$\gamma(h) = \begin{cases} 1 & h = 0 \\ -0.5 & h = \pm 2 \\ -0.25 & h = \pm 3 \\ 0 & \text{otherwise} \end{cases}$$

is the ACVF of a stationary process.

2. Suppose  $X_{1,t}$  and  $X_{2,t}$  are independent stationary processes with ACVFs  $\gamma_1(h)$  and  $\gamma_2(h)$ , and corresponding spectral density functions (SDFs)  $f_1(\lambda)$  and  $f_2(\lambda)$ , respectively.

Derive the SDF of

$$X_1 = X_{1,t} + X_{2,t}.$$

3. Derive the SDF of the stationary process  $\{X_t\}$  defined by

$$X_t = A \cos(\omega t) + B \sin(\omega t) + Z_t + 2Z_{t-1}$$

where

- $A$  and  $B$  are zero mean random variables with variance 1;
- $\omega \in (0, \pi/2)$  is a fixed constant;
- $\{Z_t\} \sim \text{WN}(0, \sigma_Z^2)$ ;
- $A, B$  and  $\{Z_t\}$  are mutually uncorrelated.

4. Suppose that

$$X_t = 0.99X_{t-3} + Z_t$$

where  $\{Z_t\} \sim \text{WN}(0, 1)$ . Find and plot the SDF for  $\{X_t\}$ , and the SDF of

$$Y_t = \frac{1}{3}(X_{t-1} + X_t + X_{t+1}).$$

5. The fractional white noise process,  $\{W_t\}$ , defined in terms of the white noise process  $\{Z_t\}$  by

$$W_t = (1 - B)^{-d} Z_t$$

for  $-0.5 < d < 0.5$  has SDF

$$f_W(\lambda) = \frac{\sigma_Z^2}{2\pi} \frac{1}{|1 - e^{-i\lambda}|^{2d}}.$$

Comment on the absolute summability of the corresponding ACVF,  $\gamma_W(h)$ .