

MATH 545: INTRODUCTION TO TIME SERIES ANALYSIS
WINTER 2025

EXERCISES 4: SOLUTIONS

1. As quoted in lectures, an absolutely summable function $\gamma(h)$ is the ACVF for a stationary process if and only if it is even and

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h) \geq 0$$

for all λ in the interval $(-\pi, \pi]$. Use this result to assess whether the function

$$\gamma(h) = \begin{cases} 1 & h = 0 \\ -0.5 & h = \pm 2 \\ -0.25 & h = \pm 3 \\ 0 & \text{otherwise} \end{cases}$$

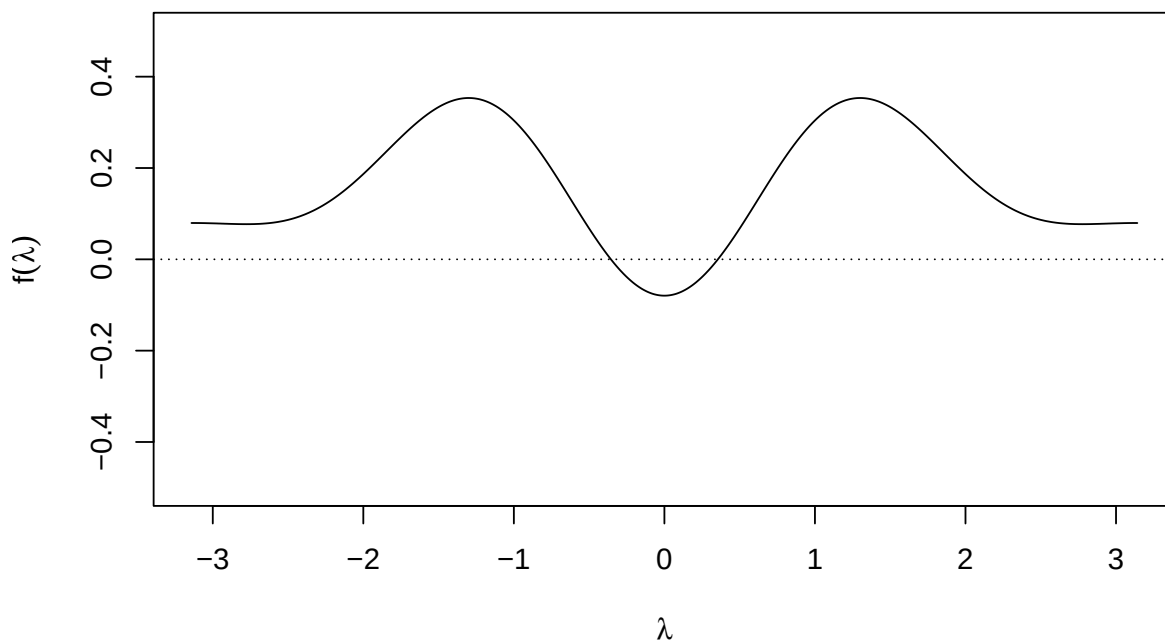
is the ACVF of a stationary process.

Solution: By direct calculation

$$\begin{aligned} f(\lambda) &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h) = \frac{1}{2\pi} \left[1 - 0.5(e^{-i2\lambda} + e^{i2\lambda}) - 0.25(e^{-i3\lambda} + e^{i3\lambda}) \right] \\ &= \frac{1}{2\pi} [1 - \cos(2\lambda) - 0.5 \cos(3\lambda)] \end{aligned}$$

This function is not non-negative as (for example) $f(0) = -0.5$. Therefore, $\gamma(\cdot)$ is not the ACVF of a stationary process.

```
lam<-seq(-pi,pi,length=1000)
flam<-(1-cos(2*lam)-0.5*cos(3*lam))/(2*pi)
par(mar=c(4,4,2,2))
plot(lam,flam,type='l',ylim=range(-0.5,0.5),
     xlab=expression(lambda),ylab=expression(f(lambda)))
abline(h=0,lty=3)
```



If instead we had

$$\gamma(h) = \begin{cases} 1 & h = 0 \\ -0.25 & h = \pm 2 \\ -0.125 & h = \pm 3 \\ 0 & \text{otherwise} \end{cases}$$

then the equivalent calculation leads to

$$f(\lambda) = \frac{1}{2\pi} [1 - 0.5 \cos(2\lambda) - 0.25 \cos(3\lambda)]$$

which is non-negative and even, so this is a valid ACVF. Using a discrete approximation, we can simulate a process with this ACVF/SDF. Let

$$\{\omega_j = \pi(j/J) \quad j = 0, \dots, J\}$$

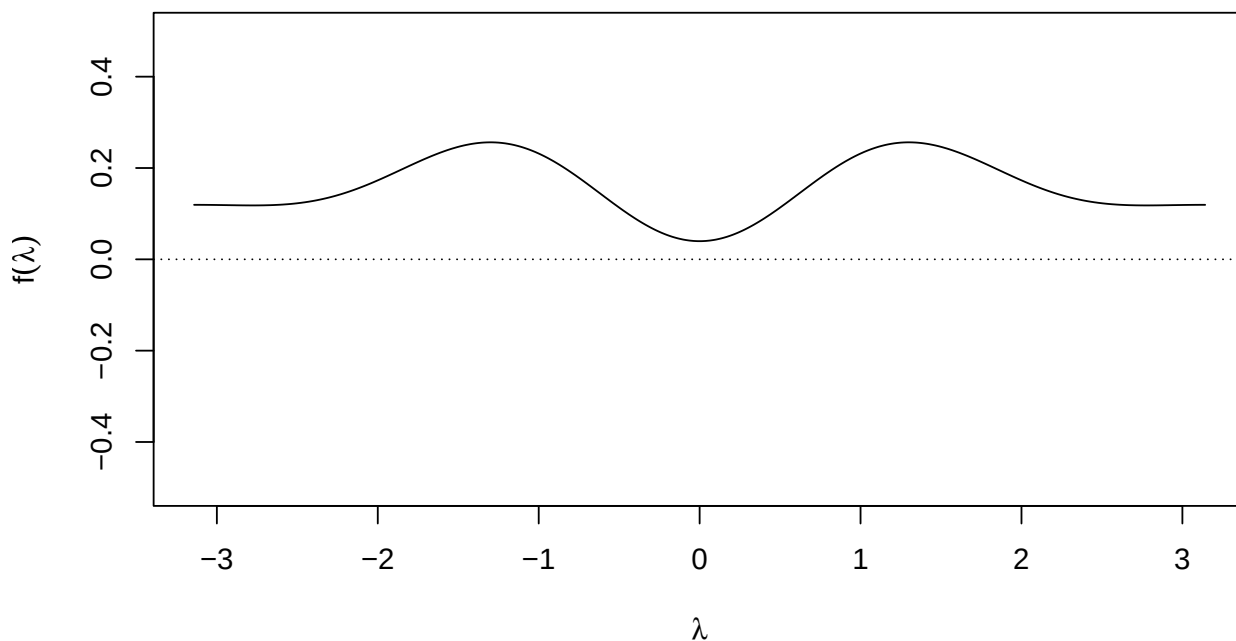
and

$$\sigma_j^2 = \frac{1}{2\pi} [1 - 0.5 \cos(2\omega_j) - 0.25 \cos(3\omega_j)]$$

and then define

$$X_t = \int_{-\pi}^{\pi} [\cos(t\lambda) + i \sin(t\lambda)] dZ(\lambda)$$

```
lam<-seq(-pi,pi,length=1000)
flam<-(1-0.5*cos(2*lam)-0.25*cos(3*lam))/(2*pi)
par(mar=c(4,4,2,0))
plot(lam,flam,type='l',ylim=range(-0.5,0.5),
      xlab=expression(lambda),ylab=expression(f(lambda)))
abline(h=0,lty=3)
```



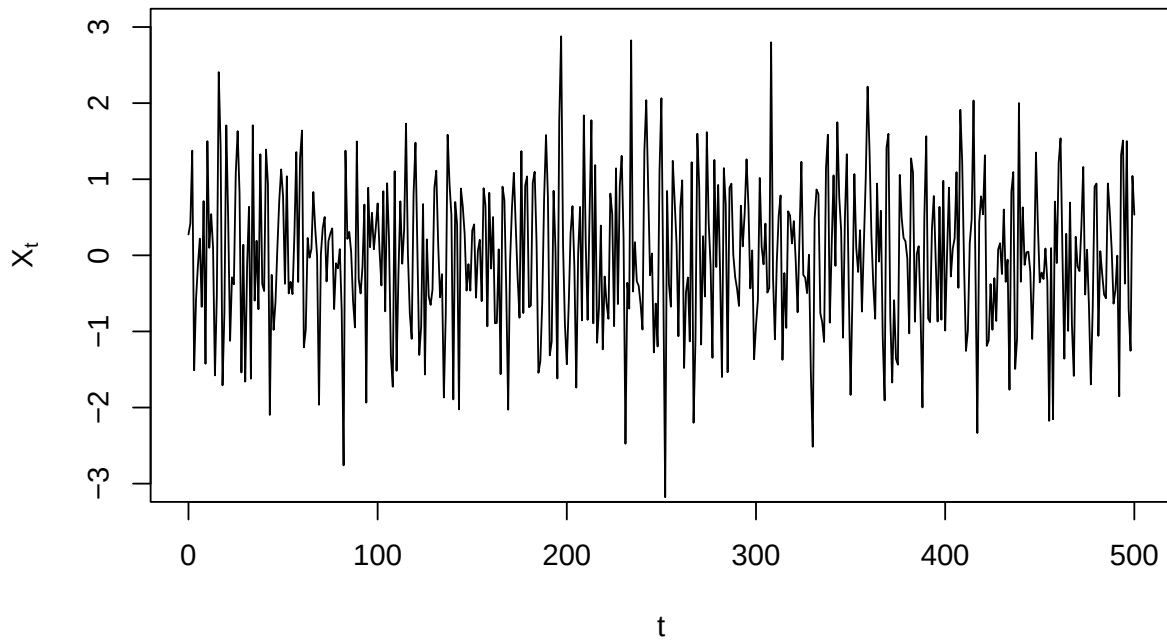
```
k<-500
lam<-seq(-pi,pi,by=0.01)
omega<-c(0:k)*pi/k
sigseq<-(1/k)*(1-0.5*cos(2*omega)-0.25*cos(3*omega))
sum(sigseq[2:k])+0.5*(sigseq[1]+sigseq[k+1])

## [1] 1

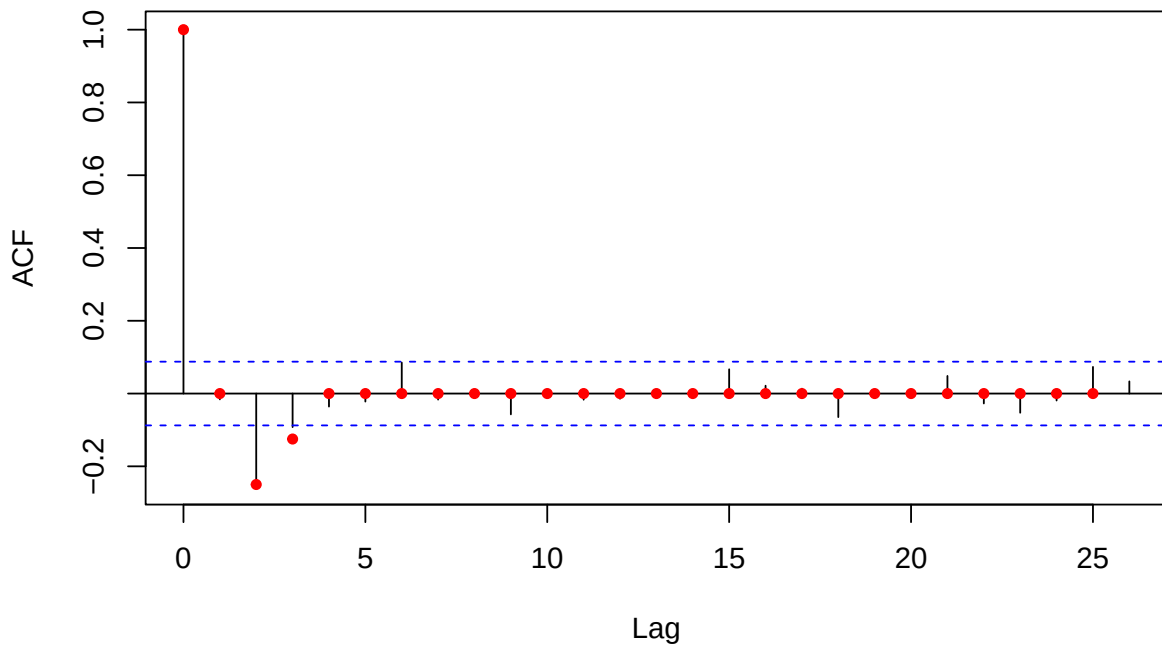
hvec<-c(0:25)
gammah<-0.5*sigseq[1]*cos(omega[1]*hvec)
for(j in 2:k){
  gammah<-gammah+sigseq[j]*cos(omega[j]*hvec)
}
gammah<-gammah+0.5*sigseq[k+1]*cos(omega[k+1]*hvec)
round(gammah,6)

## [1] 1.000 0.000 -0.250 -0.125 0.000 0.000 0.000 0.000 0.000 0.000 0.000
## [11] 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
## [21] 0.000 0.000 0.000 0.000 0.000 0.000
```

```
Uvec<-rnorm(k+1)*sqrt(sigseq)
Vvec<-rnorm(k+1)*sqrt(sigseq)
tvec<-seq(0,500,by=1)
Xt<-tvec*0
for(j in 1:(k+1)){
  Xt<-Xt+Uvec[j]*cos(omega[j]*tvec)+Vvec[j]*sin(omega[j]*tvec)
}
par(mar=c(4,4,2,2))
plot(tvec,Xt,type='l',xlab='t',ylab=expression(X[t]),ylim=range(-3,3))
```



```
acf(Xt)
points(hvec,gammah,pch=19,cex=0.7,col='red')
```



2. Suppose $X_{1,t}$ and $X_{2,t}$ are independent stationary processes with ACVFs $\gamma_1(h)$ and $\gamma_2(h)$, and corresponding spectral density functions (SDFs) $f_1(\lambda)$ and $f_2(\lambda)$, respectively.

Derive the SDF of

$$X_1 = X_{1,t} + X_{2,t}.$$

Solution: We have that

$$\gamma_X(h) = \mathbb{E}[X_t X_{t+h}] = \mathbb{E}[(X_{1,t} + X_{2,t})(X_{1,t+h} + X_{2,t+h})] = \mathbb{E}[X_{1,t} X_{1,t+h}] + \mathbb{E}[X_{2,t} X_{2,t+h}]$$

by independence. Hence

$$\gamma_X(h) = \gamma_1(h) + \gamma_2(h)$$

and

$$\begin{aligned} f_X(\lambda) &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h) \\ &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} (\gamma_1(h) + \gamma_2(h)) \\ &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma_1(h) + \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma_2(h) \\ &= f_1(\lambda) + f_2(\lambda). \end{aligned}$$

3. Derive the SDF of the stationary process $\{X_t\}$ defined by

$$X_t = A \cos(\omega t) + B \sin(\omega t) + Z_t + 2Z_{t-1}$$

where

- A and B are zero mean random variables with variance 1;
- $\omega \in (0, \pi/2)$ is a fixed constant;
- $\{Z_t\} \sim \text{WN}(0, \sigma_Z^2)$;
- A, B and $\{Z_t\}$ are mutually uncorrelated.

Solution: Using the result from Q2, with

$$f_1(\lambda) = \begin{cases} \frac{1}{2} & \lambda = -\omega \\ \frac{1}{2} & \lambda = \omega \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_2(\lambda) = \frac{\sigma_Z^2}{2\pi} |(1 + 2e^{-i\lambda})|^2 = \frac{\sigma_Z^2}{2\pi} [1 + 4(1 + \cos(\lambda))]$$

which is the SDF of the MA(1) process with $\theta = 2$. Then

$$f_X(\lambda) = f_1(\lambda) + f_2(\lambda)$$

is the SDF of $\{X_t\}$. It is an SDF of mixed type.

4. Suppose that

$$X_t = 0.99X_{t-3} + Z_t$$

where $\{Z_t\} \sim WN(0, 1)$. Find and plot the SDF for $\{X_t\}$, and the SDF of

$$Y_t = \frac{1}{3}(X_{t-1} + X_t + X_{t+1}).$$

Solution: Here we have an AR(3) process with $\phi_1 = \phi_2 = 0$. The AR polynomial is

$$\Phi(z) = 1 - 0.99z^3$$

which has reciprocal roots less than 1 in modulus so the process is stationary and has SDF

$$f_X(\lambda) = \frac{1}{2\pi} \frac{1}{|1 - 0.99e^{-i3\lambda}|^2}$$

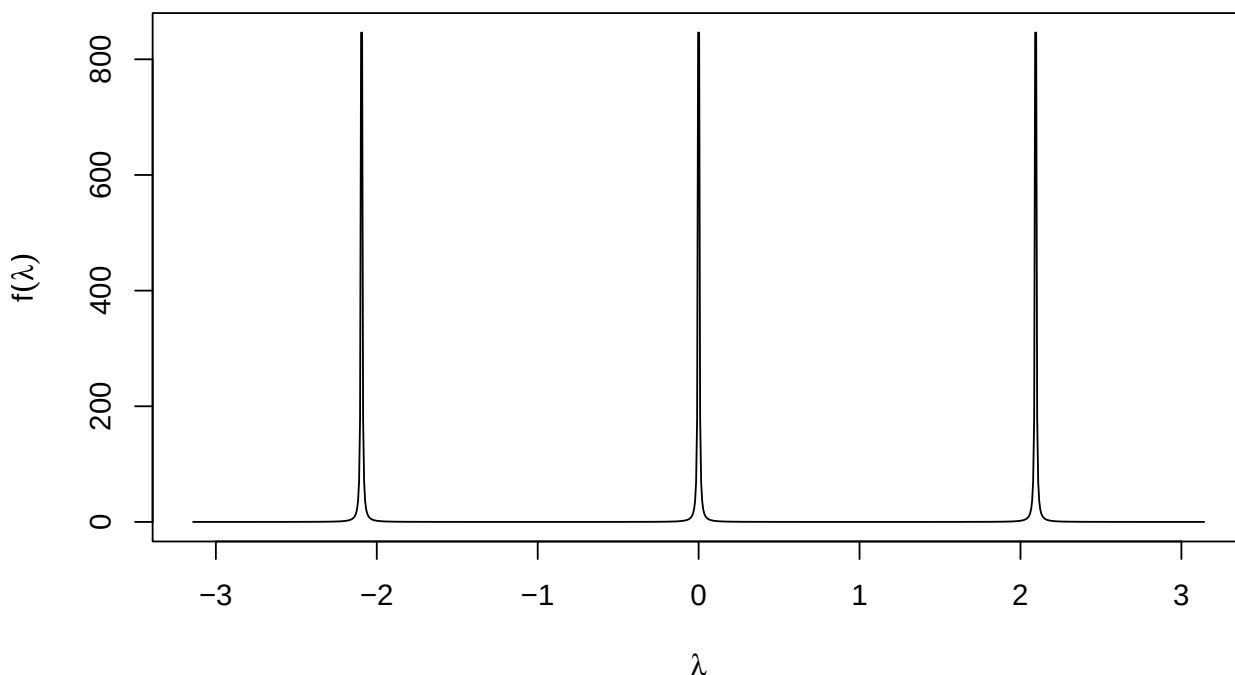
```
Mod(polyroot(c(1,0,0,-0.99)))    #Roots

## [1] 1.003356 1.003356 1.003356

1/Mod(polyroot(c(1,0,0,-0.99)))    #Reciprocal roots

## [1] 0.9966555 0.9966555 0.9966555

lam<-seq(-pi,pi,length=1000)
flam<-(1/Mod(1-0.99*exp(-1i*3*lam))^2)/(2*pi)
par(mar=c(4,4,1,0))
plot(lam,flam,type='l',xlab=expression(lambda),ylab=expression(f(lambda)))
```



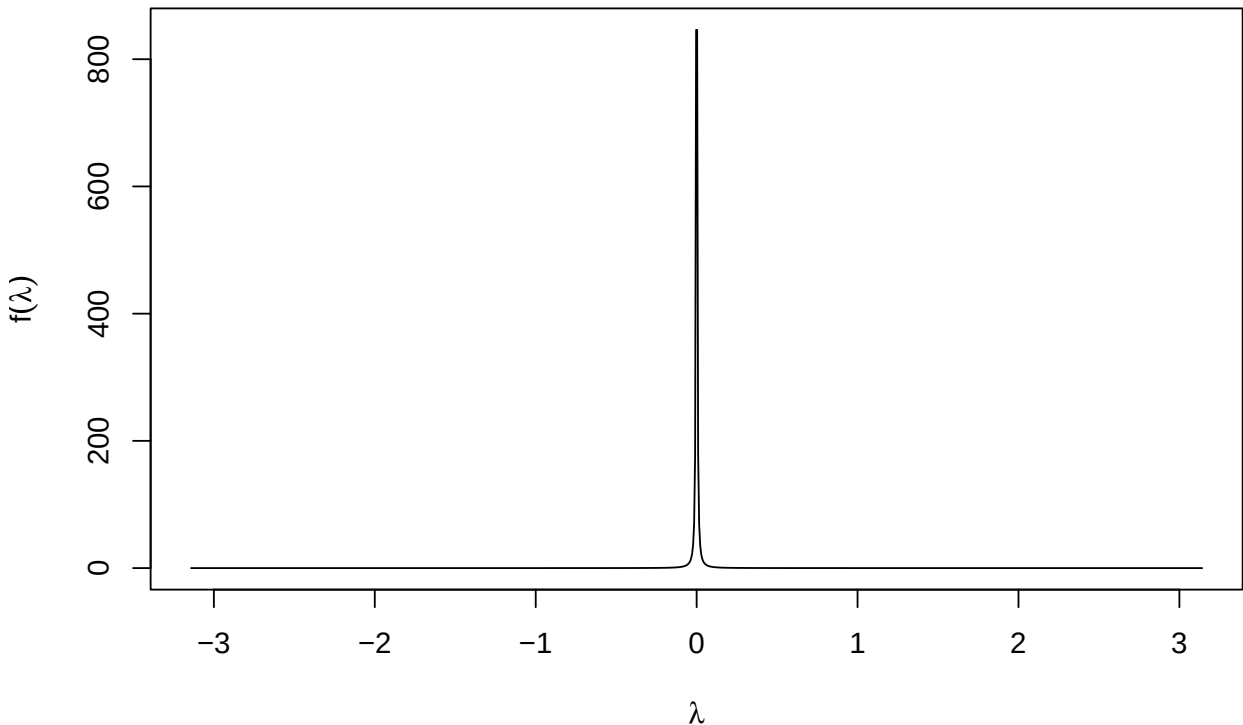
Now let $\{Y_t\}$ be the filtered version of $\{X_t\}$ given by

$$Y_t = \frac{1}{3}(X_{t-1} + X_t + X_{t+1}) = \sum_{j=-\infty}^{\infty} \psi_j X_{t-j} = \Psi(B)X_t$$

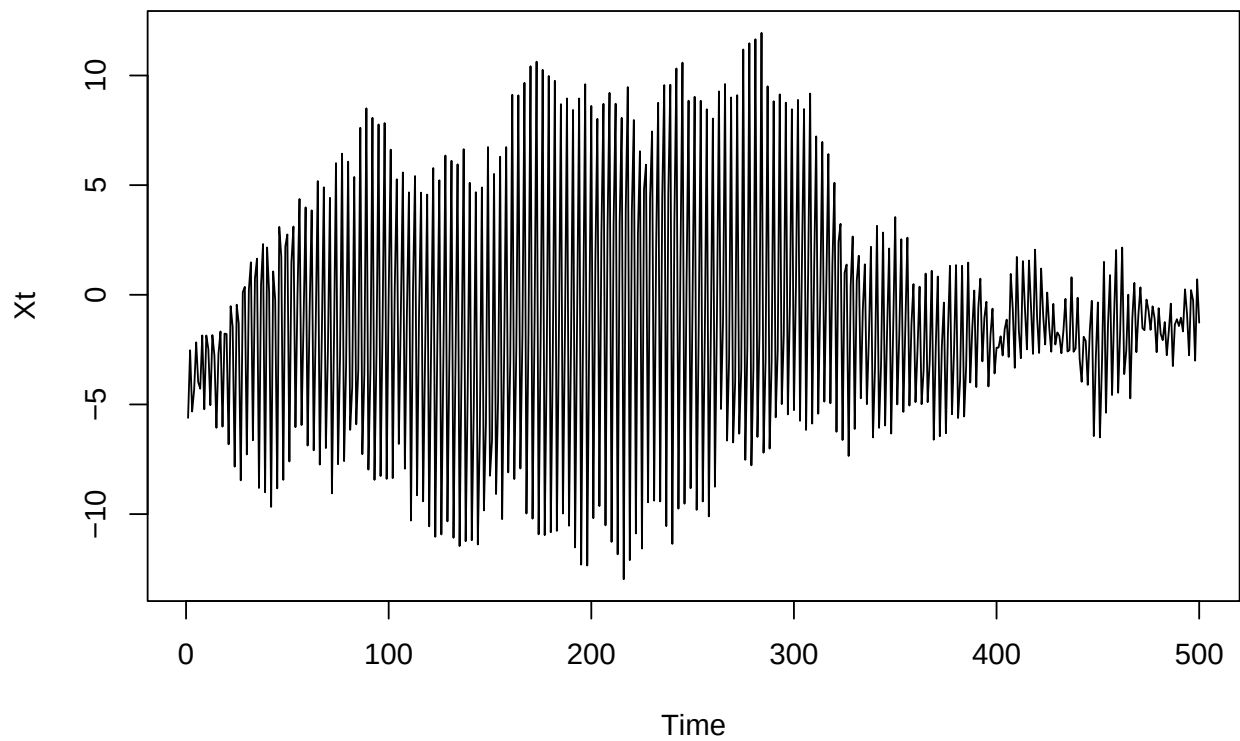
where $\psi_j = 1/3$ for $j = -1, 0, 1$, and $\psi_j = 0$ otherwise. Then

$$f_Y(\lambda) = |\Psi(e^{-i\lambda})|^2 f_X(\lambda) = \left(\frac{1}{3}\right)^2 |1 + e^{i\lambda} + e^{-i\lambda}|^2 f_X(\lambda)$$

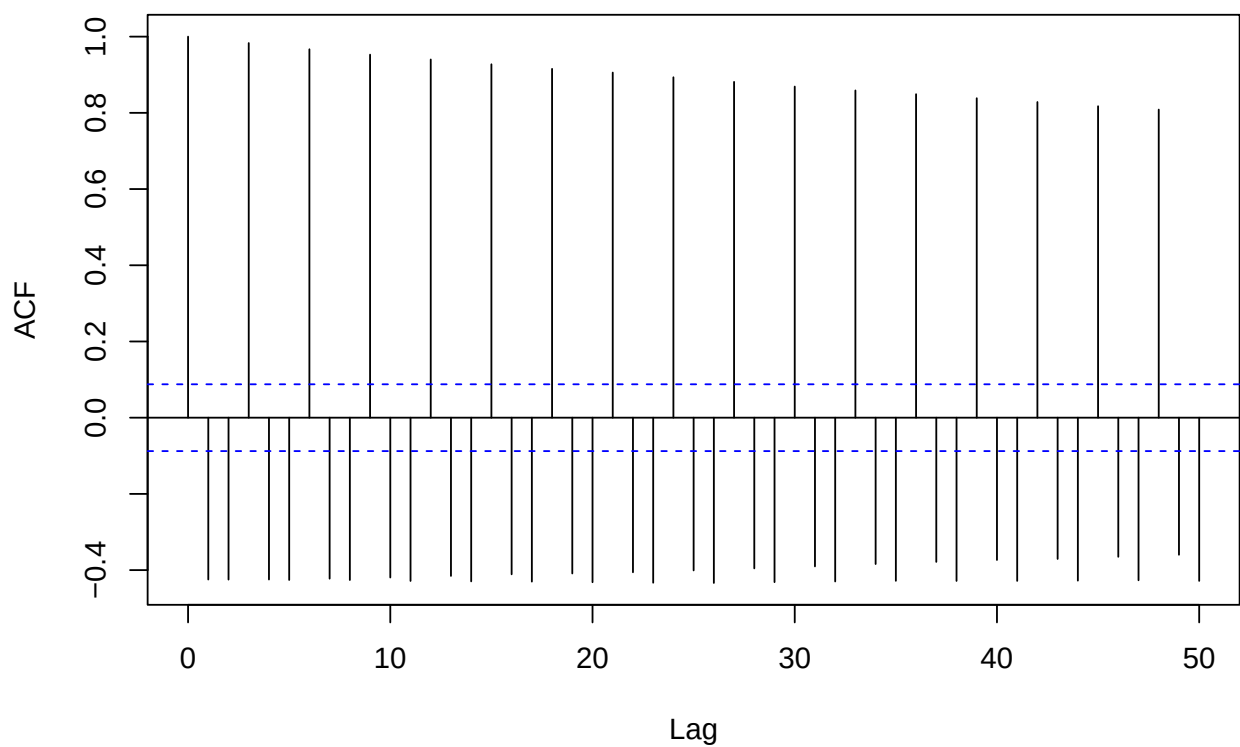
```
lam<-seq(-pi,pi,length=1000)
flam<-(1/Mod(1-0.99*exp(-1i*3*lam))^2)/(2*pi)
flam<-flam*(1/9)*Mod(1+exp(1i*lam)+exp(-1i*lam))^2
par(mar=c(4,4,2,0))
plot(lam,flam,type='l',xlab=expression(lambda),ylab=expression(f(lambda)))
```



```
set.seed(3)
n<-500
Xt<-arima.sim(n,model=list(ar=c(0,0,0.99)))
par(mar=c(4,4,1,0))
plot(Xt,type='l')
```

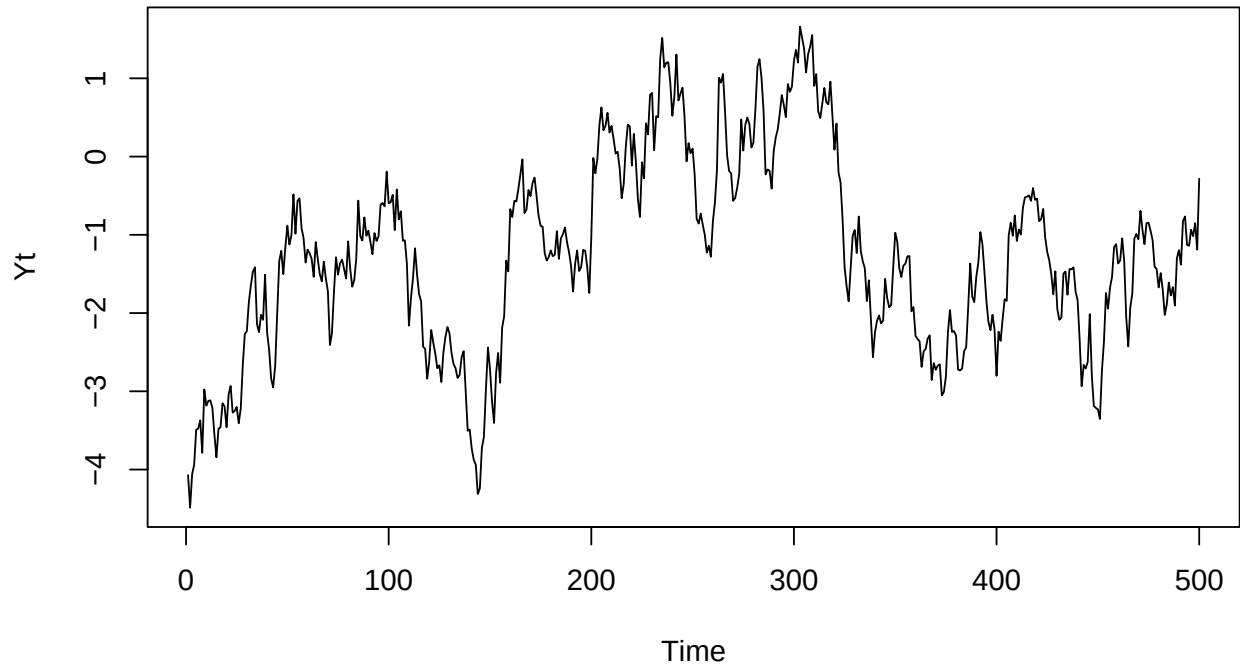


```
acf(Xt, lag.max=50)
```

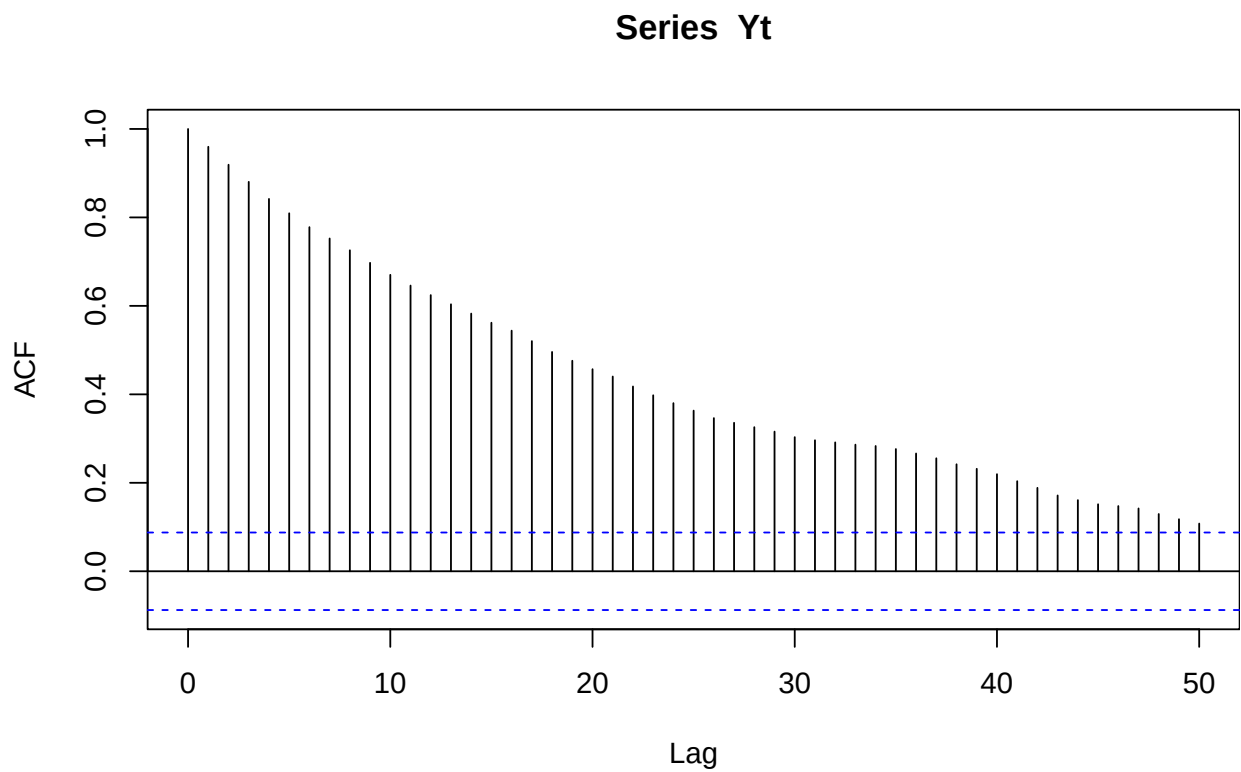


```
Yt<-Xt
Yt[1]<-(Xt[1]+Xt[2])/2; Yt[n]<-(Xt[n-1]+Xt[n])/2
for(i in 2:(n-1)){Yt[i]<-mean(Xt[c((i-1):(i+1))])}
```

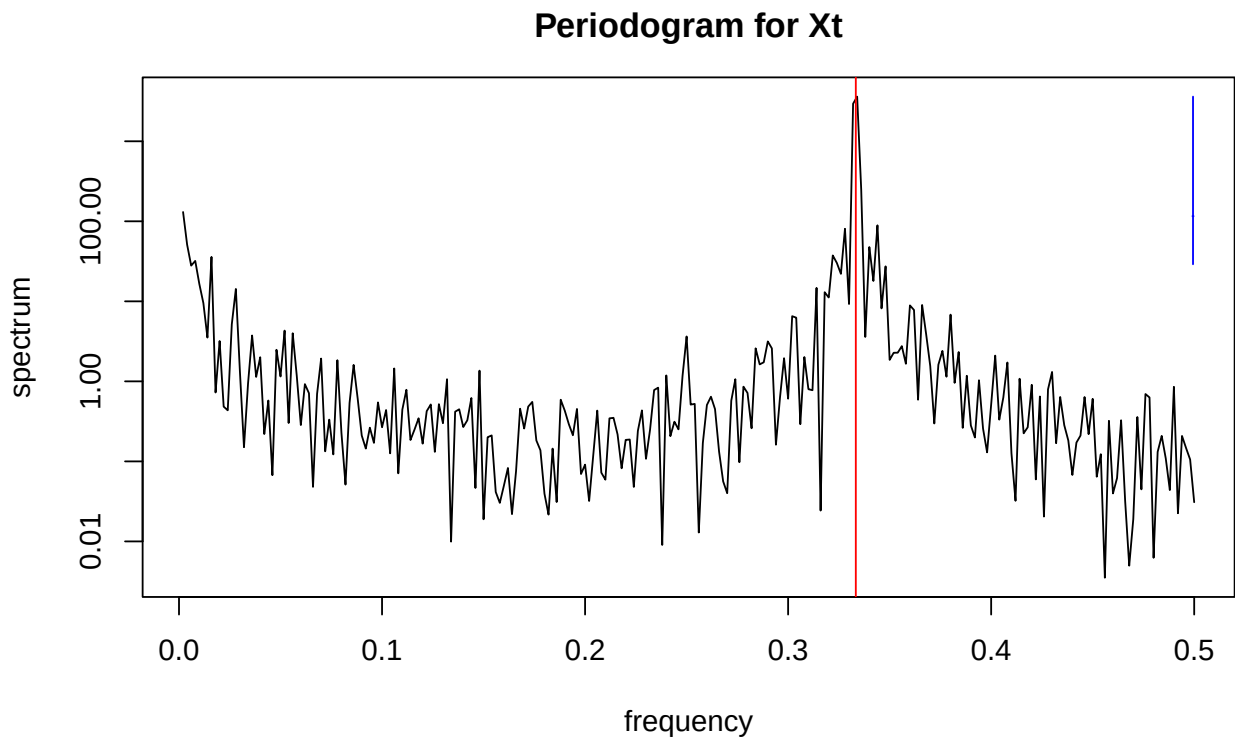
```
par(mar=c(4,4,3,0))  
plot(Yt,type='l')
```



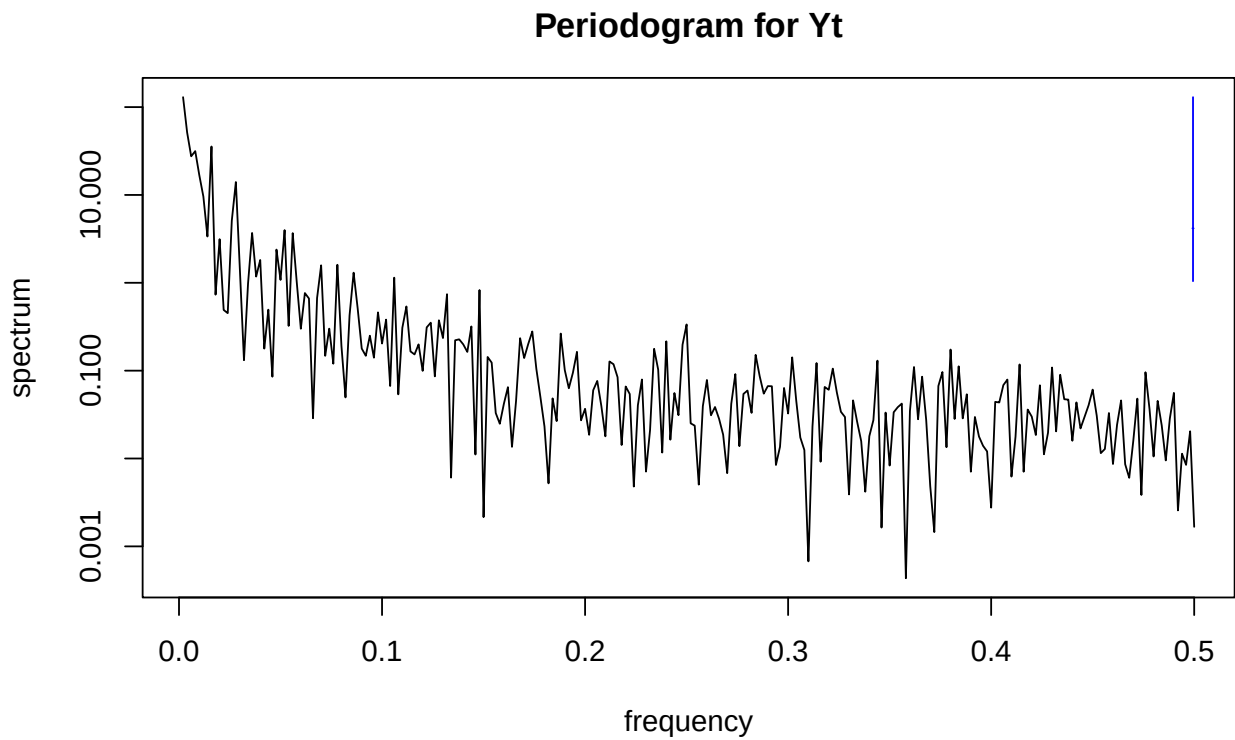
```
acf(Yt,lag.max=50)
```



```
spectrum(Xt,demean=F,taper=0,main='Periodogram for Xt')  
abline(v=1/3,col='red')
```



```
spectrum(Yt,demean=F,taper=0,main='Periodogram for Yt')
```



5. The fractional white noise process, $\{W_t\}$, defined in terms of the white noise process $\{Z_t\}$ by

$$W_t = (1 - B)^{-d} Z_t$$

for $-0.5 < d < 0.5$ has SDF

$$f_W(\lambda) = \frac{\sigma_Z^2}{2\pi} \frac{1}{|1 - e^{-i\lambda}|^{2d}}.$$

Comment on the absolute summability of the corresponding ACVF, $\gamma_W(h)$.

Solution: From lectures, we have the formula that for large $h > 0$

$$\gamma_W(h) \approx Ch^{2d-1}$$

for some constant C . The value $2d - 1$ lies in the range $(-2, 0)$, so it is evident that if $d < 0$, then $2d - 1 < -1$, and $\gamma_W(h)$ is absolutely summable, whereas if $d > 0$, $-1 < 2d - 1 < 0$ it is not; this follows as the sum

$$\sum_{h=1}^{\infty} h^{-\alpha}$$

is convergent if and only if $\alpha > 1$.

Note that for h large enough

$$\log \gamma_W(h) = \log C + (2d - 1) \log h$$

so plotting $\log \gamma_W(h)$ versus $\log h$ will lead to a straight line with slope $2d - 1$. For $d = 0.4$,

```
H<-1000
hvec<-c(1:H)
d<-0.4
gamma0<-gamma(1-2*d)/gamma(1-d^2)
gammah<-gamma0*exp(lgamma(hvec+d)+lgamma(1-d)-lgamma(hvec-d+1)-lgamma(d))
gammah[H]

## [1] 0.6899167

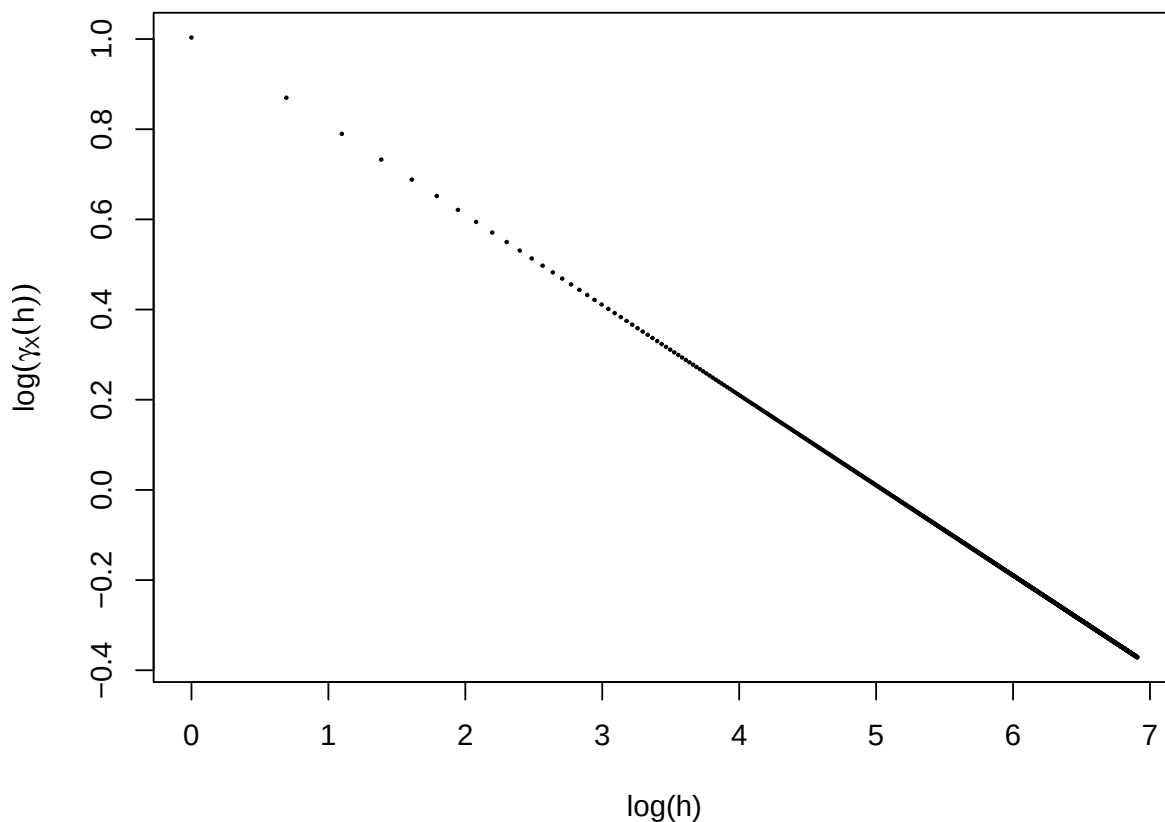
sum(abs(c(gamma0,gammah)))

## [1] 864.7864

cumsum(gammah)[100*c(1:10)]

## [1] 135.1816 236.4050 327.5507 412.7067 493.6670 571.4329 646.6385 719.7193
## [9] 790.9918 860.6953

par(mar=c(4,4,2,2))
plot(log(hvec), log(gammah),pch=19,cex=0.2,
     xlab='log(h)',ylab=expression(log(gamma[X](h))))
```



```
lm(log(gammah)~log(hvec))

##
## Call:
## lm(formula = log(gammah) ~ log(hvec))
##
## Coefficients:
## (Intercept)      log(hvec)
##      1.0100      -0.1999
```

Here the slope is -0.2 . For $d = -0.1$, the slope is -1.2 .

```
d<--0.1
gamma0<-gamma(1-2*d)/gamma(1-d^2)
gammah<-gamma0*exp(lgamma(hvec+d)+lgamma(1-d)-lgamma(hvec-d+1)-lgamma(d))
gammah[H]

## [1] 0.00002041238

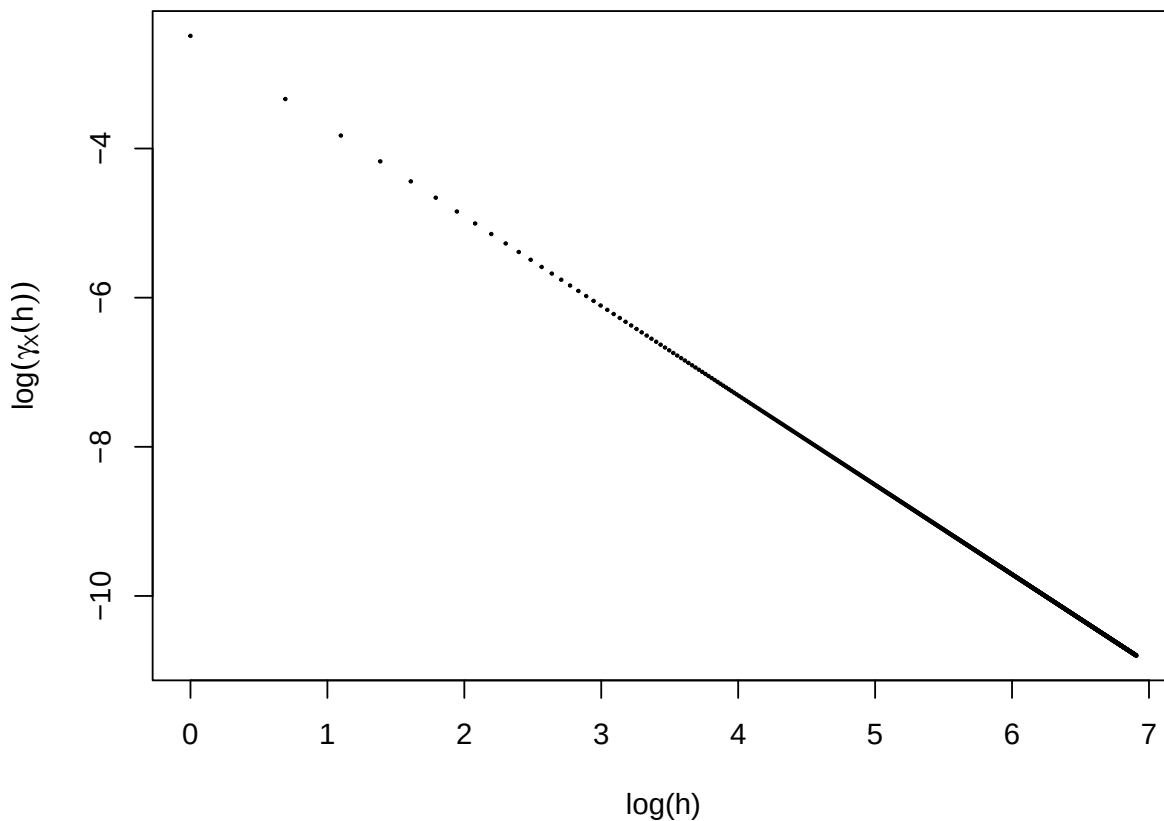
sum(abs(c(gamma0,gammah)))

## [1] 1.267161

cumsum(gammah)[100*c(1:10)]
```

```
## [1] 0.2948086 0.3156569 0.3265984 0.3338460 0.3391895 0.3433829 0.3468116
## [8] 0.3496978 0.3521806 0.3543527
```

```
par(mar=c(4,4,2,2))
plot(log(hvec), log(gammah), pch=19, cex=0.2,
     xlab='log(h)', ylab=expression(log(gamma[X](h))))
```



```
lm(log(gammah)~log(hvec))

##
## Call:
## lm(formula = log(gammah) ~ log(hvec))
##
## Coefficients:
## (Intercept)    log(hvec)
##      -2.509       -1.200
```

Therefore for $d > 0$, the ACVF converges slow enough so that the sum of its absolute values is divergent. This renders the use of the frequency-domain approach to inference based on the SDF is more palatable than time-domain approach.