

MATH 545: INTRODUCTION TO TIME SERIES ANALYSIS

WINTER 2025

EXERCISES 2: SOLUTIONS

- Find the form of the optimal forecasts for X_{n+h} from $X_1, \dots, X_n$ assumed to be from a zero mean time series process under the following specifications.

(a) $n = 50, h = 1$ for the process

$$X_t = \frac{1}{2}X_{t-1} + Z_t$$

(b) $n = 10, h = 2$, for the process

$$X_t = Z_t - \frac{1}{2}Z_{t-1}$$

(c) $n = 20, h = 5$, for the process

$$(1 - 0.9B)(1 - 1.2B + 0.32B^2)X_t = Z_t$$

Solution: The general form of the solution in the zero mean case is

$$\hat{X}_{n+h} = \sum_{i=1}^n a_i X_{n-i+1} = a_1 X_n + a_2 X_{n-1} + \dots + a_n X_1$$

where the coefficients $\mathbf{a}$ solve

$$\mathbf{\Gamma}_n \mathbf{a} = \boldsymbol{\gamma}_{h:(h+n-1)} = (\gamma_X(h), \dots, \gamma_X(h+n-1))^T$$

where

$$[\mathbf{\Gamma}_n]_{i,j} = \gamma_X(|i-j|)$$

and where we can populate left and right hand sides using the ACF after dividing through by $\gamma_X(0)$. The function `ARMAacf` can be used to compute the ACF, and the `toeplitz` function can be used to form $\mathbf{\Gamma}_n$.

(a) $n = 50, h = 1$ for the AR(1) process with parameter $\phi = 1/2$.

```
n<-50
h<-1
gamv<-ARMAacf(ar=1/2,lag.max=n+h)
Gam<-toeplitz(gamv[1:n])
gamvec<-gamv[c(h:(n+h-1))+1]
a<-solve(Gam,gamvec)
a

## [1] 0.5 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0
## [20] 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0
## [39] 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0
```

That is, we have that $a_1 = 0.5$, with $a_i = 0$ for $i = 2, \dots, n$, and the optimal forecast is

$$\hat{X}_{n+1} = \phi X_n$$

as is intuitively reasonable in the AR(1) model.

- (b) $n = 10, h = 2$, for the MA(1) process with parameter $\theta = -1/2$.

```
n<-10
h<-2
gamv<-ARMAacf(ma=-1/2,lag.max=n+h)
Gam<-toeplitz(gamv[1:n])
gamvec<-gamv[c(h:(n+h-1))+1]
a<-solve(Gam,gamvec)
a

## [1] 0 0 0 0 0 0 0 0 0 0
```

That is, we have $a_i = 0$ for $i = 1, 2, \dots, n$, and the optimal forecast is

$$\hat{X}_{n+2} = 0$$

This intuitively reasonable in the MA(1) model, as observations more than one apart in the sequence are uncorrelated.

- (c) $n = 20, h = 5$, for the AR(3) process

$$(1 - 0.9B)(1 - 0.8B)(1 - 0.4B)X_t = Z_t$$

or equivalently

$$(1 - 2.1B + 1.4B^2 - 0.288B^3)X_t = Z_t$$

that is

$$X_t = 2.1X_{t-1} - 1.4X_{t-2} + 0.288X_{t-3} + Z_t.$$

```
library(polynom)
p1<-polynomial(c(1,-0.9))
p2<-polynomial(c(1,-1.2,0.32))
p<-p1*p2
p

## 1 - 2.1*x + 1.4*x^2 - 0.288*x^3

round(polyroot(p),3)

## [1] 1.111+0i 1.250+0i 2.500+0i

round(1/polyroot(p),3)

## [1] 0.9+0i 0.8+0i 0.4+0i

ar.coef<--coef(p)[-1]
ar.coef

## [1] 2.100 -1.400 0.288

n<-20
h<-5
gamv<-ARMAacf(ar=ar.coef,lag.max=n+h)
Gam<-toeplitz(gamv[1:n])
gamvec<-gamv[c(h:(n+h-1))+1]
a<-solve(Gam,gamvec)
round(a,6)
```

##	[1]	4.331250	-4.677308	1.179562	0.000000	0.000000	0.000000	0.000000
##	[8]	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
##	[15]	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

That is, we have that

$$(a_1, a_2, a_3) = (4.33125, -4.677308, 1.1795616)$$

with $a_i = 0$ for $i = 2, \dots, n$, and the optimal forecast is

$$\hat{X}_{n+5} = 4.33125X_n - 4.677308X_{n-1} + 1.1795616X_{n-2}$$

that is, depending on the last three X values only, as is intuitively reasonable in the AR(3) model.

2. Construct the system of linear equations needed to derive the ACVF $\gamma_X(h)$ for the following stationary process

$$X_t = \frac{5}{12}X_{t-1} + \frac{1}{6}X_{t-2} + Z_t + 2Z_{t-1}$$

Solution: This is an ARMA(2, 1) system with parameters $\phi_1 = 5/12$, $\phi_2 = 1/6$ and $\theta_1 = 2$, and $p = 2$ and $q = 1$, so the rule of thumb would be to compute for $h = 0, 1, 2$ explicitly, and then use the recursion. In general we have

$$\mathbb{E}[X_t X_{t-h}] = \phi_1 \mathbb{E}[X_{t-1} X_{t-h}] + \phi_2 \mathbb{E}[X_{t-2} X_{t-h}] + \mathbb{E}[Z_t X_{t-h}] + \theta_1 \mathbb{E}[Z_{t-1} X_{t-h}]$$

We use the linear process representation of $\{X_t\}$ to compute the final two expectations. For ψ_j , we have the recursive calculation

$$\psi_j = \begin{cases} \theta_j + \sum_{k=1}^j \phi_k \psi_{j-k} & 0 \leq j < \max\{p, q+1\} \\ \sum_{k=1}^p \phi_k \psi_{j-k} & j \geq \max\{p, q+1\} \end{cases}$$

so in this case

$$\psi_j = \begin{cases} \theta_j + \sum_{k=1}^j \phi_k \psi_{j-k} & 0 \leq j < 2 \\ \sum_{k=1}^p \phi_k \psi_{j-k} & j \geq 2 \end{cases}$$

and hence, setting $\theta_0 = 1$, we have that $\psi_1 = \theta_1 + \phi_1$. Therefore we obtain the system

$$\begin{aligned} h=0: \quad \gamma_X(0) &= \phi_1 \gamma_X(1) + \phi_2 \gamma_X(2) + \sigma_Z^2(1 + \theta_1 \psi_1) \\ h=1: \quad \gamma_X(1) &= \phi_1 \gamma_X(0) + \phi_2 \gamma_X(1) + \sigma_Z^2 \theta_1 \\ h=2: \quad \gamma_X(2) &= \phi_1 \gamma_X(1) + \phi_2 \gamma_X(0) \\ h=3: \quad \gamma_X(3) &= \phi_1 \gamma_X(2) + \phi_2 \gamma_X(1) \end{aligned}$$

and we can solve the first three equations as a linear system, and then use the fourth equation as a recursion for $h \geq 3$. Therefore we have the system

$$\begin{bmatrix} 1 & -\phi_1 & -\phi_2 \\ -\phi_1 & 1-\phi_2 & 0 \\ -\phi_2 & -\phi_1 & 1 \end{bmatrix} \begin{bmatrix} \gamma_X(0) \\ \gamma_X(1) \\ \gamma_X(2) \end{bmatrix} = \begin{bmatrix} \sigma_Z^2(1 + \theta_1 \psi_1) \\ \sigma_Z^2 \theta_1 \\ 0 \end{bmatrix}$$

```

#Check for causal
covec<-c(1,-5/12,-1/6)
polyroot(covec)

+ [1] 1.5+0i -4.0+0i

1/polyroot(covec)

+ [1] 0.6666667+0i -0.2500000+0i

p<-2;q<-1;m<-max(p,q+1);lmax<-200
phi.vec<-theta.vec<-rep(0,lmax+1)
phi.vec[1:p]<-c(5/12,1/6)
theta.vec[1:q]<-c(2)
true.psi.vec<-ARMAtoMA(ar=phi.vec[1:p],ma=theta.vec[1:q],lag.max=lmax) #True values
psi.vec<-rep(0,lmax+1)
psi.vec[0+1]<-1
for(j in 1:lmax){
  kvec<-1:min(j,p)
  psi.vec[j+1]<-theta.vec[j]+sum(phi.vec[kvec]*psi.vec[j-kvec+1])
}
cbind(1:10,psi.vec[1:10+1],true.psi.vec[1:10]) #Psi vector

+      [,1]      [,2]      [,3]
+ [1,] 1 2.41666667 2.41666667
+ [2,] 2 1.17361111 1.17361111
+ [3,] 3 0.89178241 0.89178241
+ [4,] 4 0.56717785 0.56717785
+ [5,] 5 0.38495451 0.38495451
+ [6,] 6 0.25492735 0.25492735
+ [7,] 7 0.17037882 0.17037882
+ [8,] 8 0.11347907 0.11347907
+ [9,] 9 0.07567941 0.07567941
+ [10,] 10 0.05044627 0.05044627

#True values of ACF
true.rhoX<-ARMAacf(ar=phi.vec[1:p],ma=theta.vec[1:q],lag.max=lmax)
#Solving the linear system
Amat<-matrix(0,3,3)
Amat[1,]<-c(1,-phi.vec[1:p])
Amat[2,]<-c(-phi.vec[1],1-phi.vec[2],0)
Amat[3,]<-c(-phi.vec[2],-phi.vec[1],1)
bvec<-c(1+theta.vec[1]*psi.vec[1+1], theta.vec[1],0);gammaX<-solve(Amat,bvec)
gammaX<-c(gammaX,phi.vec[1]*gammaX[3]+phi.vec[2]*gammaX[2])
rbind(gammaX/gammaX[1],true.rhoX[1:4])

+      0      1      2      3
+ [1,] 1 0.75 0.4791667 0.3246528
+ [2,] 1 0.75 0.4791667 0.3246528

```

Thus the explicit solution computed via the system of equations matches the ACF given by ARMAacf out to lag $h = 3$.

3. Find the form of the autocovariance generating function (ACVGF) for the following stationary processes:

(a) For $-1 < \phi < 1$,

$$X_t = \phi X_{t-1} + Z_t$$

(b) For real parameters θ_1, θ_2 ,

$$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}$$

(c)

$$\left(1 + B + \frac{1}{2}B^2\right) X_t = Z_t + Z_{t-1} + Z_{t-2} + Z_{t-3}.$$

Leave your answer as a ratio of polynomials.

Solution: The ACVGF is defined generally by

$$G_X(z) = \sum_{h=-\infty}^{\infty} \gamma_X(h) z^h \quad 1/r < |z| < r$$

whenever this sum is convergent for some r . In the stationary ARMA class of models

$$\Phi(B)X_t = \Theta(B)Z_t$$

we saw that

$$G_X(z) = \sigma_Z^2 \frac{\Theta(z)}{\Phi(z)} \frac{\Theta(z^{-1})}{\Phi(z^{-1})}$$

(a) Here $\Phi(z) = 1 - \phi z$ and $\Theta(z) = 1$, so

$$G_X(z) = \sigma_Z^2 \frac{1}{(1 - \phi z)} \frac{1}{(1 - \phi/z)} = \frac{\sigma_Z^2}{(1 + \phi) - (\phi + 1/\phi)z}$$

(b) Here $\Phi(z) = 1$ and $\Theta(z) = 1 + \theta_1 z + \theta_2 z^2$, so

$$G_X(z) = \sigma_Z^2 (1 + \theta_1 z + \theta_2 z^2)(1 + \theta_1 z^{-1} + \theta_2 z^{-2})$$

(c) Here $\Phi(z) = (1 - \xi_1 z)(1 - \xi_2 z)$ where (ξ_1, ξ_2) are the reciprocal roots of

$$1 + z + \frac{1}{2}z^2 = 0$$

and $\Theta(z) = 1 + z + z^2 + z^3$. Note that the roots of the AR polynomial are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 2}}{1} = -1 \pm i$$

with modulus $\sqrt{2} > 1$, and the reciprocal roots are $(-1 \pm i)/2$.

```
t1<-polynomial(c(1,1,1/2))
polyroot(t1)

## [1] -1+1i -1-1i

1/polyroot(t1)

## [1] -0.5-0.5i -0.5+0.5i
```

Hence

$$G_X(z) = \sigma_Z^2 \frac{(1+z+z^2+z^3)}{(1+z+z^2/2)} \frac{(1+z^{-1}+z^{-2}+z^{-3})}{(1+z^{-1}+z^{-2}/2)}$$

or, multiplying through in the second terms by z^3 in numerator and denominator

$$G_X(z) = \sigma_Z^2 \frac{(1+z+z^2+z^3)}{(1+z+z^2/2)} \frac{(1+z+z^2+z^3)}{(z/2+z^2+z^3)}$$

4. The ARMA(1, 1) process defined as the solution to the equation

$$X_t = \frac{6}{5}X_{t-1} + Z_t + 2Z_{t-1}$$

is neither causal nor invertible. Find a stationary, causal and invertible process defined with respect to a different white noise process $\{Z_t^*\}$ that has the same ACVF as $\{X_t\}$.

Solution: Here

$$(1 - (6/5)B) X_t = (1 + 2B)Z_t$$

so we consider the white noise process

$$Z_t^* = \{\Theta^*(B)\}^{-1} \Phi^*(B) X_t$$

where

$$\Theta^*(z) = (1 + (1/2)z) \quad \Phi^*(z) = (1 - (5/6)z).$$

We have from results in lectures that $\{Z_t^*\}$ is white noise with variance

$$\sigma_{Z^*}^2 = \sigma_Z^2 \frac{2^2}{(6/5)^2} = \frac{100}{36} \sigma_Z^2$$

and the process

$$(1 - (5/6)B)X_t = (1 + (1/2)B)Z_t^*$$

is a causal and invertible representation of $\{X_t\}$.