

MATH 545: INTRODUCTION TO TIME SERIES ANALYSIS

WINTER 2025

EXERCISES 1: SOLUTIONS

Not for assessment.

1. Find the autocovariance function (ACVF), $\gamma_X(h)$, for the process $\{X_t\}$ defined by

$$X_t = Z_t + 0.7Z_{t-1} - 0.2Z_{t-2} \quad t \in \mathbb{Z}$$

Here, for $h = 0, 1, 2, \dots$,

$$\gamma_X(h) = \text{Cov}[X_t, X_{t+h}] = \mathbb{E}[X_t X_{t+h}] = \mathbb{E}[X_{t-h} X_t].$$

Solution: We have for $h \geq 0$

$$\begin{aligned} \mathbb{E}[X_t X_{t+h}] &= \mathbb{E}[(Z_t + 0.7Z_{t-1} - 0.2Z_{t-2})(Z_{t+h} + 0.7Z_{t+h-1} - 0.2Z_{t+h-2})] \\ &= \mathbb{E}[Z_t Z_{t+h}] + 0.7\mathbb{E}[Z_t Z_{t+h-1}] - 0.2\mathbb{E}[Z_t Z_{t+h-2}] \\ &\quad + 0.7\mathbb{E}[Z_{t-1} Z_{t+h}] + 0.49\mathbb{E}[Z_{t-1} Z_{t+h-1}] - 0.14\mathbb{E}[Z_{t-1} Z_{t+h-2}] \\ &\quad - 0.2\mathbb{E}[Z_{t-2} Z_{t+h}] - 0.14\mathbb{E}[Z_{t-2} Z_{t+h-1}] + 0.04\mathbb{E}[Z_{t-2} Z_{t+h-2}]. \end{aligned}$$

By the white noise property of $\{Z_t\}$, $\mathbb{E}[Z_t Z_s] = 0$ unless $t = s$. Therefore we see that $\mathbb{E}[X_t X_{t+h}]$ is zero if $h > 2$, as for such h the indices can never be made equal.

$h = 0$	$\mathbb{E}[X_t^2] = (1.00 + 0.49 + 0.04)\sigma_Z^2$	1st, 5th and 9th terms
$h = 1$	$\mathbb{E}[X_t X_{t+1}] = (0.7 - 0.14)\sigma_Z^2$	2nd and 6th terms
$h = 2$	$\mathbb{E}[X_t X_{t+2}] = -0.2\sigma_Z^2$	3rd term

Hence

$$\gamma_X(h) = \begin{cases} 1.53\sigma_Z^2 & h = 0 \\ 0.56\sigma_Z^2 & h = \pm 1 \\ -0.2\sigma_Z^2 & h = \pm 2 \\ 0 & h \geq 3 \end{cases}$$

This model is the MA(2) process, and the ACVF can be computed in R using the library `ltsa` and the function `tacvfARMA`:

```
library(ltsa)
gamma.X<-tacvfARMA(phi = numeric(0), theta = c(-0.7,0.2), maxLag = 4, sigma2 = 1)
gamma.X

## [1] 1.53 0.56 -0.20 0.00 0.00
```

The ACF $\rho_X(h)$ can be computed using the `ARMAacf` function

```
ARMAacf(ma = c(0.7,-0.2), lag.max = 4)

##           0           1           2           3           4
##  1.0000000  0.3660131 -0.1307190  0.0000000  0.0000000

gamma.X/gamma.X[1]                                #Check

## [1]  1.0000000  0.3660131 -0.1307190  0.0000000  0.0000000
```

Note the different sign conventions for the two coefficients in the two function calls.

2. Identify whether each of the following numerical ACVFs, labelled (a), (b), (c) and (d),

h	$\gamma_X(h)$			
	(a)	(b)	(c)	(d)
0	1.0100	2.0200	1.21333	0.66667
1	-0.1000	-0.2000	0.50667	0.33333
2	0.0000	0.0000	0.25333	0.16667
3	0.0000	0.0000	0.12667	0.08333
4	0.0000	0.0000	0.06333	0.04167

correspond to

- a moving average, or MA(1), process,

$$X_t = Z_t + \theta_1 Z_{t-1}$$

for some θ_1 to be identified.

- an autoregressive, or AR(1) process

$$X_t = \phi_1 X_{t-1} + Z_t$$

for some ϕ_1 to be identified, where $|\phi_1| < 1$.

- some other process.

You may assume stationarity. The numerical values are rounded to four decimal places, and reported up to lag $h = 4$.

Solution: For (a) and (b), we see that the ACVF is zero past lag 1. Therefore, the MA(1) model is appropriate, and using the previously established forms for the MA(1) ACVF, we know that

$$\gamma_X(0) = \sigma_Z^2(1 + \theta_1)^2 \quad \gamma_X(1) = \theta_1 \sigma_Z^2$$

so we can identify for (a) that $\theta_1 = -0.1$ and $\sigma_Z^2 = 1$, whereas for (b) $\theta_1 = -0.1$ and $\sigma_Z^2 = 2$.

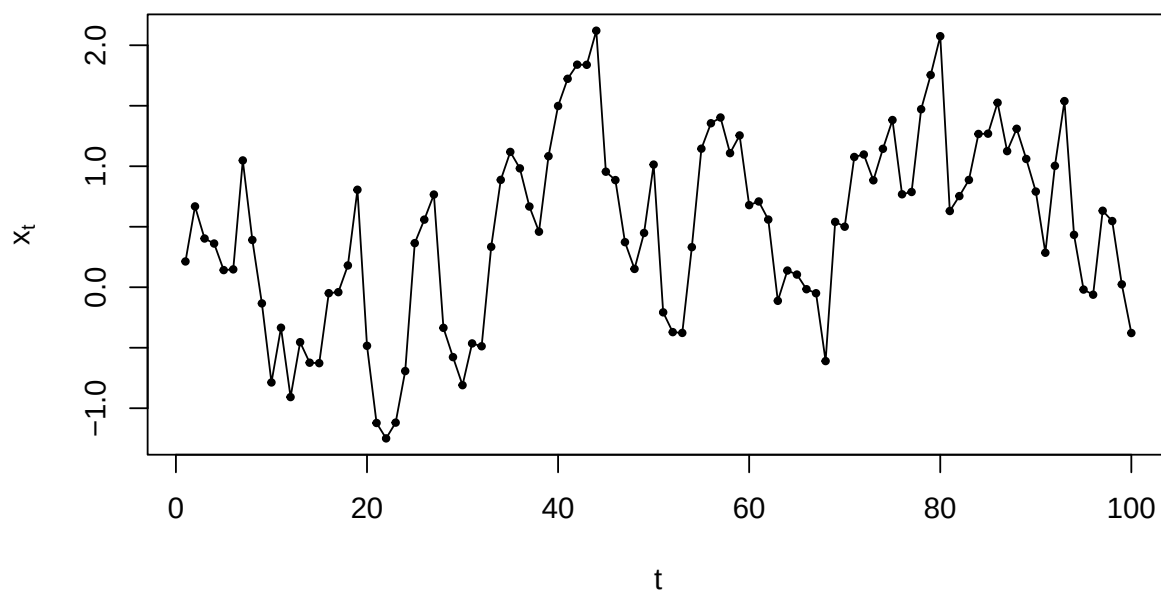
For (d), we observe a geometric decay of ACV values, with multiplier term $\phi_1 = 0.5$ (i.e. the ACV values drop by one half as h increases by 1.. We also know from notes that

$$\gamma_X(0) = \frac{\sigma_Z^2}{1 - \phi_1^2}$$

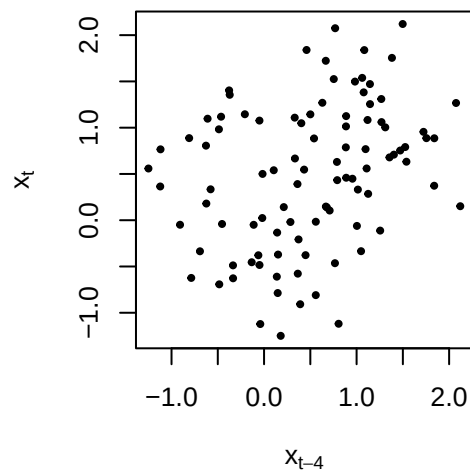
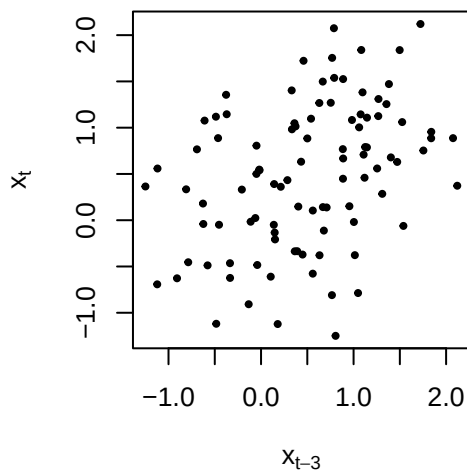
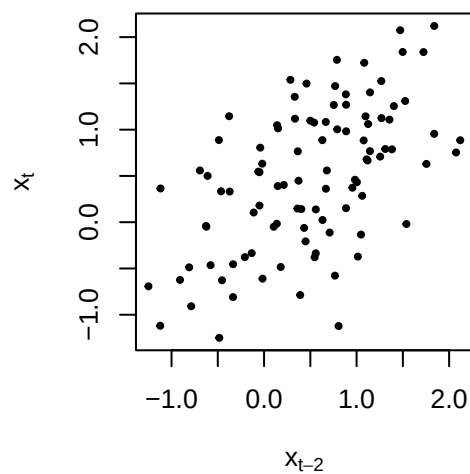
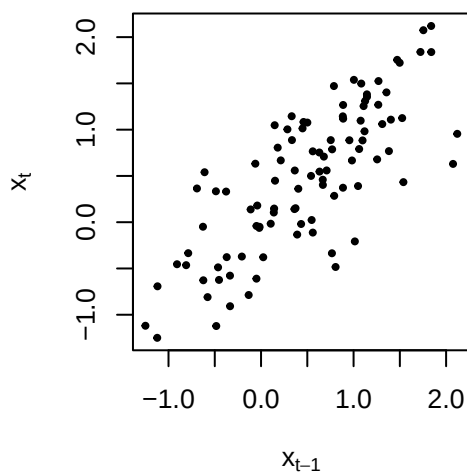
so may conclude that $\sigma_Z^2 = 0.5$.

For (c), we have neither the MA(1) ACVF nor the AR(1) ACVF, as neither of the characteristic behaviours described above are present.

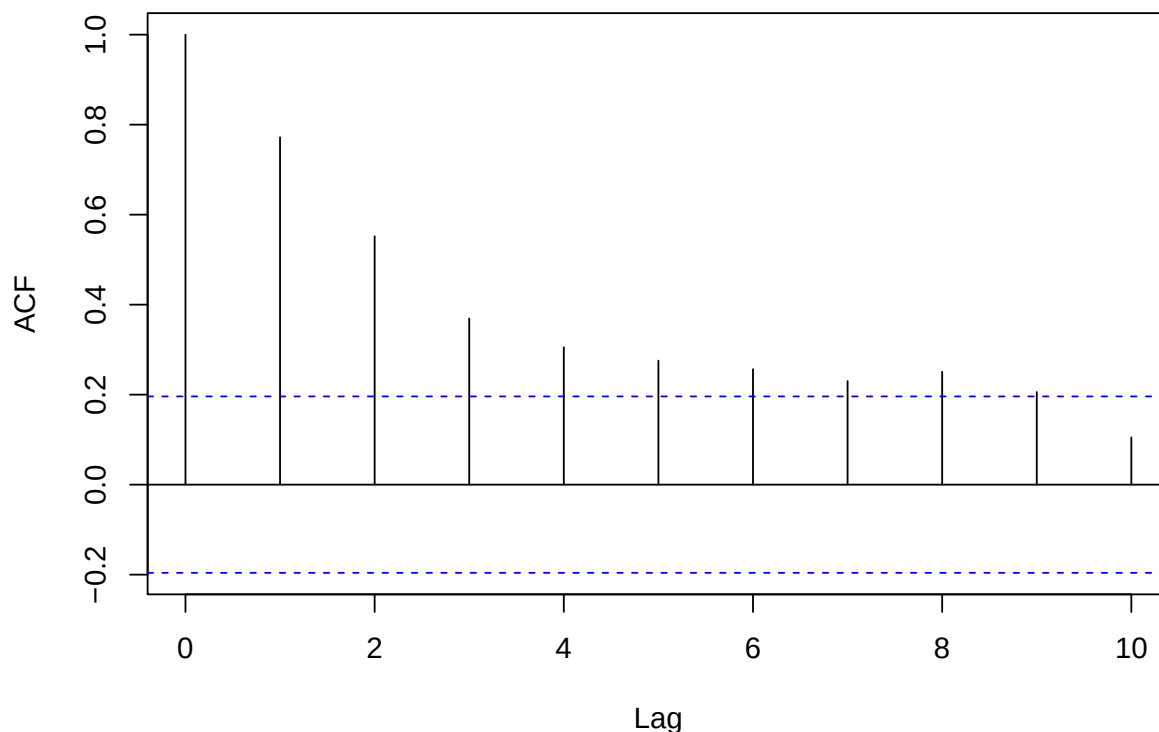
3. The following plots depict features of an observed time series, denoted $x_t, t = 1, \dots, 100$.



These scatterplots depict the relationship between x_t and x_{t-h} for $h = 1, 2, 3, 4$.



Finally, this is a plot of the *estimated* autocorrelation function (ACF), $\hat{\rho}_X(h)$, based on the 100 data points.



The blue dotted horizontal lines in this plot indicate statistical thresholds (i.e. critical values) for tests of the hypotheses

$$H_0 : \rho_X(h) = 0$$

$$H_a : \rho_X(h) \neq 0$$

for each h .

On the basis of this empirical information, suggest a time series model that may be appropriate as the data generating model for these data.

Solution: The positive dependence between successive x_t values, and the geometric decay in the estimated ACF values, indicate that an AR(1) model may be the data generating model. From the ACF plot, it seems that as the estimated value at lag $h = 1$ is around 0.75, so therefore it appears that the value of ϕ_1 is approximately 0.75.