

MATH 545 - ASSIGNMENT 3 : SOLUTIONS

1. Compute, using R or otherwise, the optimal linear one-step-ahead forecast for X_{n+1} based the following data set of $n = 11$ data points

-0.93 -0.89 -0.63 -0.38 0.76 0.03 -0.56 -0.19 -0.11 -0.82 -0.49

assuming the following models:

- (a) The AR(1) model: $X_t = 0.9X_{t-1} + Z_t$; 2 MARKS
 (b) The ARMA(1, 1) model: $X_t = 0.5X_{t-1} + Z_t - Z_{t-1}$; 3 MARKS
 (c) The ARMA(2, 1) model: $X_t = 0.2X_{t-1} - 0.5X_{t-2} + Z_t + Z_{t-1}$. 3 MARKS

Give the numerical value of the forecast, and the code used. Here, it is not necessary to use the recursive algorithms for matrix inversion as the matrix inversion function `solve` (or equivalent) can be used directly.

Solution: We solve the linear system $\Gamma_X(n)\mathbf{a} = \gamma_X(n, 1)$ with $n = 11$, that is

$$\begin{bmatrix} \gamma_X(0) & \gamma_X(1) & \gamma_X(2) & \cdots & \gamma_X(n-1) \\ \gamma_X(1) & \gamma_X(0) & \gamma_X(1) & \cdots & \gamma_X(n-2) \\ \gamma_X(2) & \gamma_X(1) & \gamma_X(0) & \cdots & \gamma_X(n-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_X(n-1) & \gamma_X(n-2) & \gamma_X(n-3) & \cdots & \gamma_X(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \gamma_X(1) \\ \gamma_X(2) \\ \gamma_X(3) \\ \vdots \\ \gamma_X(n) \end{bmatrix}$$

using the notation from lectures. Then, the optimal forecast is

$$\hat{X}_{n+1} = \sum_{i=1}^n a_i X_{n+i-1}.$$

Note that we can formulate the linear system after dividing both sides by $\gamma_X(0)$, so in fact we only need the ACF $\rho_X(h)$, $h = 0, 1, 2, \dots$

```
x<-c(-0.93,-0.89,-0.63,-0.38,0.76,0.03,-0.56,-0.19,-0.11,-0.82,-0.49)
n<-11
```

- (a) The AR(1) model: $X_t = 0.9X_{t-1} + Z_t$;

```
GammaXn<-toeplitz(ARMAacf(ar=0.9,lag.max=n-1))
gammaXn1<-ARMAacf(ar=0.9,lag.max=n)[-1]
a<-solve(GammaXn,gammaXn1)
a
+ [1] 0.9 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0
xhat<-sum(a*rev(x))
xhat
+ [1] -0.441
```

The forecast is -0.441. Note that this solution for a confirms the result from lectures that the optimal forecast for the AR(1) model is $\hat{X}_{n+1} = \phi X_n$.

2 MARKS

(b) The ARMA(1,1) model: $X_t = 0.5X_{t-1} + Z_t - Z_{t-1}$;

```
GammaXn<-toeplitz(ARMAacf(ar=0.5,ma=-1,lag.max=n-1))
gammaXn1<-ARMAacf(ar=0.5,ma=-1,lag.max=n)[-1]
a<-solve(GammaXn,gammaXn1)
round(a,3)

+ [1] -0.429 -0.393 -0.357 -0.321 -0.286 -0.250 -0.214 -0.179 -0.143 -0.107
+ [11] -0.071

xhat<-sum(a*rev(x))
xhat

+ [1] 0.9417857
```

The forecast is 0.9417857.

3 MARKS

(c) The ARMA(2,1) model: $X_t = 0.2X_{t-1} - 0.5X_{t-2} + Z_t + Z_{t-1}$.

```
GammaXn<-toeplitz(ARMAacf(ar=c(0.2,-0.5),ma=1,lag.max=n-1))
gammaXn1<-ARMAacf(ar=c(0.2,-0.5),ma=1,lag.max=n)[-1]
a<-solve(GammaXn,gammaXn1)
round(a,3)

+ [1] 1.106 -1.492 1.332 -1.171 1.011 -0.850 0.689 -0.529 0.368 -0.208
+ [11] 0.094

xhat<-sum(a*rev(x))
xhat

+ [1] 0.7565111
```

The forecast is 0.7565111.

3 MARKS

We can check using the forecasting tools in R, although you must show the working given above.

```
as.numeric(predict(arima(x,order=c(1,0,0),fixed=c(0.9,0)),1)$pred)

+ [1] -0.441

as.numeric(predict(arima(x,order=c(1,0,1),fixed=c(0.5,-1,0)),1)$pred)

+ [1] 0.9417857

as.numeric(predict(arima(x,order=c(2,0,1),fixed=c(0.2,-0.5,1,0)),1)$pred)

+ [1] 0.7565111
```

2. Consider the model for stationary process $\{X_t\}$ defined by

$$X_t = Z_t + \theta_1 Z_{t-1} + \theta_{12} Z_{t-12}. \quad (1)$$

(a) Compute the form of the ACVF, $\gamma_X(h)$.

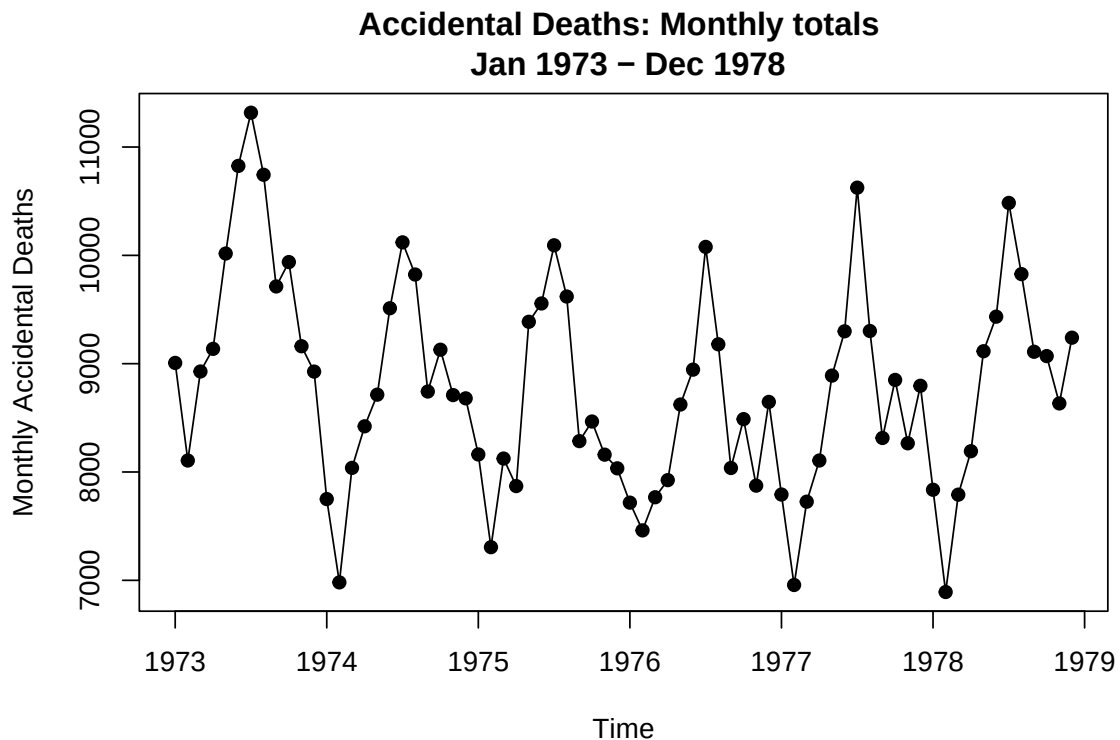
4 MARKS

(b) The data `USAccDeaths` available in R below are stationary in mean (mean estimated to be 8788.792, with no trend), but also exhibit periodic and non-stationary behaviour. After differencing successively at lag 1 and lag 12 (using the difference operators ∇ then ∇_{12}), the periodic and non-stationary behaviour can be removed. That is, denoting the original data process $\{Y_t\}$ and the differenced data process $\{X_t\}$, we define

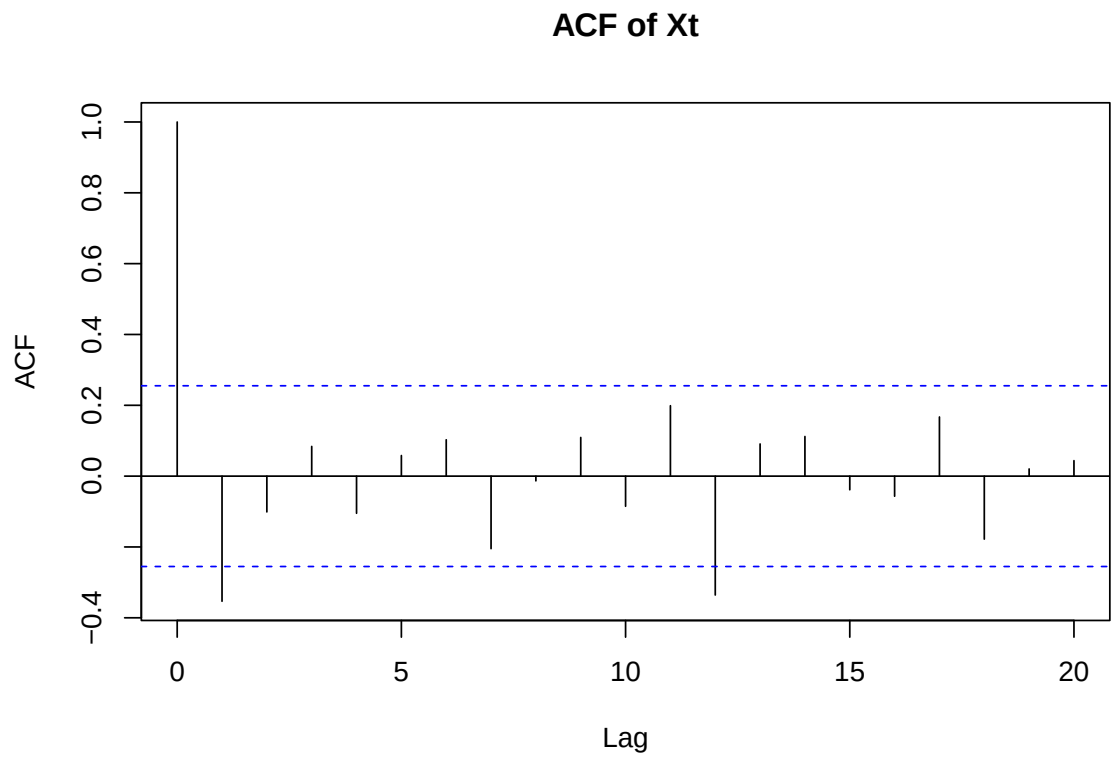
$$X_t = \nabla_{12} \nabla Y_t = (1 - B^{12})(1 - B)Y_t$$

and study $\{X_t\}$.

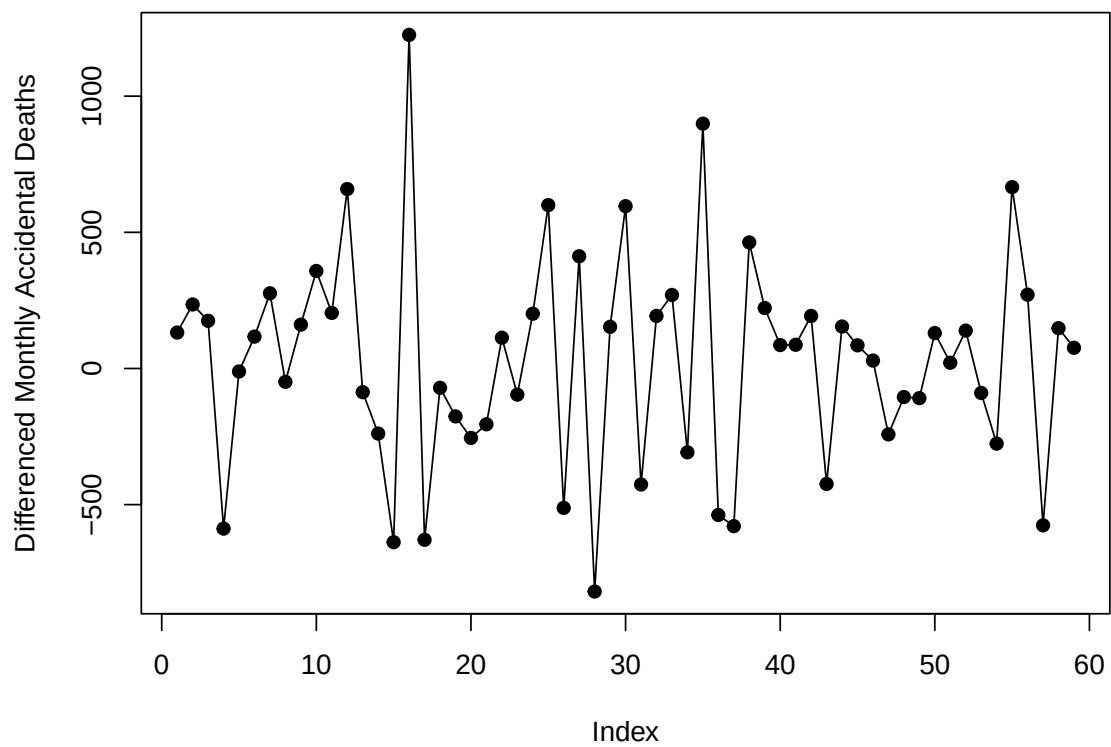
```
par(mar=c(4, 4, 3, 2))
plot(USAccDeaths, ylab="Monthly Accidental Deaths")
points(USAccDeaths, pch=19)
title("Accidental Deaths: Monthly totals \n Jan 1973 - Dec 1978")
```



```
Yt<-as.numeric(USAccDeaths); Xt<-diff(diff(Yt), lag=12)
par(mar=c(4, 4, 3, 2))
acf(Xt, lag.max=20, main='ACF of Xt')
```



```
par(mar=c(4, 4, 3, 2))
plot(Xt, type='l', ylab="Differenced Monthly Accidental Deaths")
points(Xt, pch=19)
```



Using the differenced data X_t , estimate the parameters θ_1 and θ_{12} in the model (1) when used for these data by equating theoretical and sample values of the ACF. 6 MARKS

- (c) Explain how to compute the one-step-ahead forecasts for $\{X_t\}$ and $\{Y_t\}$ based on the available data (you do not need to compute the forecasts). 2 MARKS

Solutions:

- (a) We have that $\{X_t\}$ is zero mean with variance $\sigma_Z^2(1 + \theta_1^2 + \theta_{12}^2)$. We have the usual linear process representation with $\psi_0 = 1$, $\psi_1 = \theta_1$ and $\psi_{12} = \theta_{12}$. Thus using the general formula

$$\gamma_X(h) = \sigma_Z^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}$$

we have non-zero ACVF only at lags 0, 1, 11 and 12, specifically

$$\gamma_X(h) = \begin{cases} \sigma_Z^2(1 + \theta_1^2 + \theta_{12}^2) & h = 0 \\ \sigma_Z^2\theta_1 & h = \pm 1 \\ \sigma_Z^2\theta_1\theta_{12} & h = \pm 11 \\ \sigma_Z^2\theta_{12} & h = \pm 12 \\ 0 & \text{otherwise} \end{cases}$$

4 MARKS

- (b) We have

```
Xt<-diff(diff(USAccDeaths),lag=12)
n<-length(Xt)
varX<-sum((Xt-mean(Xt))^2)/n
varX
+ [1] 151910.8
var(Xt)
+ [1] 154529.9
```

Therefore the variance of the differenced series is 154529.9018118; this uses the formula

$$\frac{1}{n-1} \sum_{i=1}^n (x_t - \bar{x})^2.$$

However, as computed in `acf` an alternative estimate is 151910.7509336 by the formula

$$\frac{1}{n} \sum_{i=1}^n (x_t - \bar{x})^2.$$

```

par(mar=c(4,4,4,1))
acf.US<-acf(Xt,lag.max=24,plot=FALSE)
rho.mat<-cbind(0:24,as.numeric(acf.US$acf))
rownames(rho.mat)<-rep(' ',25)
colnames(rho.mat)<-c('h','rho.hat')
rho.mat[1:15,]

+   h      rho.hat
+   0  1.00000000
+   1 -0.35337231
+   2 -0.10109563
+   3  0.08392623
+   4 -0.10489176
+   5  0.05819082
+   6  0.10274494
+   7 -0.20482412
+   8 -0.01338705
+   9  0.10926346
+  10 -0.08514495
+  11  0.19886656
+  12 -0.33574229
+  13  0.09087780
+  14  0.11190157

```

The only significant ACF values appear to be at $h = 1$ (or $h = 1/12$ of one year) for which the value is -0.3533723 and at lag $h = 12$ (or $h = 1$ year) for which the value is -0.3357423. Assuming the model above for these data, we must solve

$$\sigma_Z^2(1 + \hat{\theta}_1^2 + \hat{\theta}_{12}^2) = \hat{\gamma}_X(0) = 151910.7509336$$

$$\frac{\hat{\theta}_1}{(1 + \hat{\theta}_1^2 + \hat{\theta}_{12}^2)} = \hat{\rho}_X(1) = -0.3533723$$

$$\frac{\hat{\theta}_1 \hat{\theta}_{12}}{(1 + \hat{\theta}_1^2 + \hat{\theta}_{12}^2)} = \hat{\rho}_X(11) = 0.1988666$$

$$\frac{\hat{\theta}_{12}}{(1 + \hat{\theta}_1^2 + \hat{\theta}_{12}^2)} = \hat{\rho}_X(12) = -0.3357423$$

Here we have four equations in three unknowns, so for this moment system we can select two equations and solve for θ_1 and θ_{12} . Dividing the third equation by the second,

$$\hat{\theta}_{12} = \frac{\hat{\rho}_X(11)}{\hat{\rho}_X(1)} = -0.5627678.$$

Similarly, dividing the third equation by the fourth,

$$\hat{\theta}_1 = \frac{\hat{\rho}_X(11)}{\hat{\rho}_X(12)} = -0.5923191$$

so that finally

$$\hat{\sigma}_Z^2 = \frac{\hat{\gamma}_X(0)}{(1 + \hat{\theta}_1^2 + \hat{\theta}_{12}^2)} = 91098.1971677.$$

```
rho1<-as.numeric(acf.US$acf)[2]
rho11<-as.numeric(acf.US$acf)[12]
rho12<-as.numeric(acf.US$acf)[13]
theta1<-rho11/rho12; theta12<-rho11/rho1
c(theta1,theta12)

+ [1] -0.5923191 -0.5627678

sigmaZsq<-varX/(1+theta1^2+theta12^2)
sigmaZsq

+ [1] 91098.2
```

This yields

$$\frac{\hat{\theta}_1 \hat{\theta}_{12}}{(1 + \hat{\theta}_1^2 + \hat{\theta}_{12}^2)} = 0.199897.$$

which is close to the observed version of $\hat{\rho}_X(11)$. To check the result, we can compute the theoretical ACF with $(\theta_1, \theta_{12}) = (\hat{\theta}_1, \hat{\theta}_{12})$.

```
theta.vec<-rep(0,12)
theta.vec[1]<-theta1;theta.vec[12]<-theta12
theoretical.acf<-ARMAacf(ma=theta.vec,lag.max=12)
rho.mat<-cbind(0:12,theoretical.acf)
rownames(rho.mat)<-rep(' ',13)
colnames(rho.mat)<-c('h','rho')
rho.mat

+   h      rho
+  0 1.0000000
+  1 -0.3552033
+  2  0.0000000
+  3  0.0000000
+  4  0.0000000
+  5  0.0000000
+  6  0.0000000
+  7  0.0000000
+  8  0.0000000
+  9  0.0000000
+ 10  0.0000000
+ 11  0.1998970
+ 12 -0.3374819
```

- (c) For X_t , as this process is presumed stationary, we can forecast using the standard method of optimal linear forecasting based on $\Gamma_X(n)\mathbf{a} = \gamma_X(n, 1)$. For Y_t , we have that

$$X_t = (1 - B^{12})(1 - B)Y_t = (1 - B^{12})(Y_t - Y_{t-1}) = Y_t - Y_{t-12} - Y_{t-1} + Y_{t-13}.$$

so therefore

$$Y_t = X_t + Y_{t-12} + Y_{t-1} - Y_{t-13}.$$

so we may compute the forecasts for Y_t by forecasting X_t , and then using this recursive formula, assuming that $\hat{Y}_t \equiv Y_t$ for $1 \leq t \leq n$. We should recall that the original series has a non-zero mean, however, when seeding the recursion; it is plausible that the mean is constant across time.

4 MARKS

For forecasting $X_{n+1}, X_{n+2}, \dots$, we can use the standard approach. Then, for example,

$$\hat{Y}_{n+h} = \hat{X}_{n+h} + \hat{Y}_{n+h-12} + \hat{Y}_{n+h-1} - \hat{Y}_{n+h-13}$$

where $\hat{Y}_t \equiv Y_t$ for $1 \leq t \leq n$.

```
n<-length(Xt)
hvec<-1:20
xpred<-hvec*0
ypred<-c(Yt[14:72],0*hvec)
GammaXn<-toeplitz(ARMAacf(ma=theta.vec,lag.max=n-1))
for(h in hvec){
  gammaXn1<-ARMAacf(ma=theta.vec,lag.max=n+h-1)
  gammaXn1<-gammaXn1[h:(n+h-1)]
  a<-solve(GammaXn,gammaXn1)
  xhat<-sum(a*rev(Xt))
  xpred[h]<-xhat
  ypred[n+h]<-xhat+ypred[n+h-12]+ypred[n+h-1]-ypred[n+h-13]
}
par(mar=c(4,5,2,2))
plot(14:92,ypred,pch=19,cex=0.8,xlab='t',ylab=expression(hat(Y)[t]))
lines(14:92,ypred)
```

