

MATH 545 - ASSIGNMENT 2 : SOLUTIONS

1. For each of the following stationary processes, $\{X_t\}$, defined via their model equations in terms of $\{Z_t\} \sim \text{WN}(0, 1)$, find the corresponding linear process representation

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$$

that is, identify the sequence $\{\psi_j\}$. Show your derivation. If no such linear process representation exists, explain why.

(a) $X_t = Z_t + Z_{t-1} + Z_{t-4}$. 2 MARKS

(b) $X_t = \frac{1}{4}X_{t-2} + Z_t$. 3 MARKS

(c) $X_t = \frac{3}{4}X_{t-1} + Z_t + \frac{1}{2}Z_{t-1}$. 3 MARKS

Solution:

- (a) This is an MA(4) process, and by direct inspection we can see that

$$\psi_j = \begin{cases} 1 & j = 0 \\ 1 & j = 1 \\ 1 & j = 4 \\ 0 & \text{otherwise} \end{cases}$$

We can use the `ARMAtoMA` function to compute the coefficients: note that this function does not return $\psi_0 = 1$ for any ARMA process.

```
psimat<-rbind(ARMAtoMA(ma=c(1,0,0,1),lag.max=8))
colnames(psimat)<-paste('j=',1:8,sep='')
rownames(psimat)<-c('psi.j')
psimat
```

```
##      j=1 j=2 j=3 j=4 j=5 j=6 j=7 j=8
## psi.j  1  0  0  1  0  0  0  0
```

2 MARKS

- (b) Here we have

$$\left(1 - \frac{1}{4}B^2\right) X_t = Z_t$$

so the corresponding polynomial is

$$\Phi(z) = \left(1 - \frac{1}{4}z^2\right) = \left(1 - \frac{1}{2}z\right) \left(1 + \frac{1}{2}z\right)$$

Hence to find the $\{\psi_j\}$ sequence, we need to compute

$$\left(1 - \frac{1}{2}z\right)^{-1} \left(1 + \frac{1}{2}z\right)^{-1} = \left(\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k z^k\right) \left(\sum_{l=0}^{\infty} \left(-\frac{1}{2}\right)^l z^l\right) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (-1)^l \left(\frac{1}{2}\right)^{k+l} z^{k+l}$$

To identify the coefficient of z^j , fix $k + l = j$, so that $l = j - k$. Then the coefficient of ψ_j is

$$\psi_j = \sum_{k=0}^j (-1)^{j-k} \left(\frac{1}{2}\right)^j = \begin{cases} \left(\frac{1}{2}\right)^j & j = 0, 2, 4, \dots \\ 0 & j = 1, 3, 5, \dots \end{cases}$$

```
psimat<-rbind(ARMAtoMA(ar=c(0,1/4),lag.max=8))
colnames(psimat)<-paste('j=',1:8,sep='')
rownames(psimat)<-c('psi.j')
psimat
##          j=1  j=2  j=3      j=4  j=5      j=6  j=7      j=8
## psi.j    0 0.25  0 0.0625  0 0.015625  0 0.00390625
```

3 MARKS

(c) Here we have the ARMA(1,1) model

$$\left(1 - \frac{3}{4}B\right)X_t = \left(1 + \frac{1}{2}B\right)Z_t$$

and therefore need to compute the coefficients in the polynomial

$$\Psi(z) = \frac{(1 + z/2)}{(1 - 3z/4)} = \left(1 + \frac{1}{2}z\right) \left(1 + \frac{3}{4}z + \left(\frac{3}{4}\right)^2 z^2 + \dots\right)$$

Collecting terms, we see that $\psi_0 = 1$, $\psi_1 = (1/2) + (3/4) = 5/4$, and for $j \geq 2$

$$\psi_j = \left(\frac{1}{2} + \frac{3}{4}\right) \left(\frac{3}{4}\right)^{j-1} = \left(\frac{5}{4}\right) \left(\frac{3}{4}\right)^{j-1}$$

3 MARKS

```
psimat<-rbind(ARMAtoMA(ar=c(3/4),ma=c(1/2),lag.max=8))
colnames(psimat)<-paste('j=',1:8,sep='')
rownames(psimat)<-c('psi.j')
psimat
##          j=1  j=2      j=3      j=4      j=5      j=6      j=7      j=8
## psi.j 1.25 0.9375 0.703125 0.5273438 0.3955078 0.2966309 0.2224731 0.1668549
(5/4)*(3/4)^c(0:7)
## [1] 1.2500000 0.9375000 0.7031250 0.5273438 0.3955078 0.2966309 0.2224731
## [8] 0.1668549
```

2. State, with justification, whether the AR(2) process defined in terms of $\{Z_t\} \sim WN(0,1)$ via the equation

$$X_t = X_{t-1} - \frac{1}{2}X_{t-2} + Z_t \quad t \in \mathbb{Z}$$

is stationary and causal.

4 MARKS

Solution: The AR polynomial is

$$\Phi(z) = 1 - z + \frac{1}{2}z^2 = (1 - \xi_1 z)(1 - \xi_2 z)$$

where ξ_1 and ξ_2 satisfy

$$\xi_1 + \xi_2 = 1 \quad \xi_1 \xi_2 = \frac{1}{2}$$

and are either both entirely real, or a complex conjugate pair. Using the formula for the roots of a quadratic equation, we see that the roots are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 - 2}}{1} = 1 \pm i.$$

as $a = 1/2$, $b = -1$, $c = 1$ in the polynomial $\Phi(z)$. The modulus of the roots is then

$$|1 + i| = \sqrt{1 + 1} = \sqrt{2} > 1$$

and therefore $|\xi_1| = |\xi_2| = 1/\sqrt{2} < 1$ and the process is stationary and causal.

4 MARKS

In fact, the reciprocal roots are

$$\xi_1 = \frac{1}{1+i} = \frac{1}{2} - \frac{1}{2}i \quad \xi_2 = \frac{1}{1-i} = \frac{1}{2} + \frac{1}{2}i.$$

```
library(polynom)
(Phi<-polynomial(c(1,-1, 0.5)))

## 1 - x + 0.5*x^2

round(polyroot(Phi), 2)

## [1] 1+1i 1-1i

round(1/polyroot(Phi), 2)

## [1] 0.5-0.5i 0.5+0.5i
```

3. Suppose that $\{X_t\}$ is weakly stationary. Is the process $\{Y_t\}$ defined by

$$Y_t = X_{-t} \quad t \in \mathbb{Z}$$

weakly stationary? Justify your answer.

4 MARKS

Solution: If $\{X_t\}$ is weakly stationary with mean μ and variance σ^2 , then for all t

$$\mathbb{E}[Y_t] = \mathbb{E}[X_{-t}] = \mu \quad \text{Var}[Y_t] = \text{Var}[X_{-t}] = \sigma^2.$$

Also, we have

$$\mathbb{E}[Y_t Y_{t+h}] = \mathbb{E}[X_{-t} X_{-(t+h)}] = \gamma_X(h)$$

say, which does not depend on t . Therefore $\{X_t\}$ and $\{Y_t\}$ have exactly the same mean, variance, and autocovariance as each other, and $\{Y_t\}$ is weakly stationary.

4 MARKS

4. Suppose that $\{X_{1,t}\}$ and $\{X_{2,t}\}$ are independent processes (that is, $X_{1,t}$ and $X_{2,s}$ are independent for all t and s), with

$$X_{1,t} \sim \text{AR}(1) \quad X_{2,t} \sim \text{AR}(1)$$

defined with respect to two independent white noise processes $\{Z_{1,t}\} \sim \text{WN}(0, 1)$ and $\{Z_{2,t}\} \sim \text{WN}(0, 1)$. Suppose specifically that

$$X_{1,t} = \phi X_{1,t-1} + Z_{1,t}$$

$$X_{2,t} = \varphi X_{2,t-1} + Z_{2,t}$$

where $0 < \phi, \varphi < 1$, with the usual assumption that

$$\mathbb{E}[X_{1,t} Z_{1,s}] = \mathbb{E}[X_{2,t} Z_{2,s}] = 0 \quad s > t.$$

Is the process $\{Y_t\}$ defined by

$$Y_t = X_{1,t} + X_{2,t} \quad t \in \mathbb{Z}$$

stationary and causal? Justify your answer.

4 MARKS

Solution: It is immediately evident that

$$\begin{aligned}
 \mathbb{E}[Y_t] &= \mathbb{E}[X_{1,t} + X_{2,t}] = 0 \\
 \mathbb{E}[Y_t Y_{t+h}] &= \mathbb{E}[(X_{1,t} + X_{2,t})(X_{1,t+h} + X_{2,t+h})] \\
 &= \mathbb{E}[X_{1,t} X_{1,t+h}] + \mathbb{E}[X_{2,t} X_{2,t+h}] + \mathbb{E}[X_{1,t} X_{2,t+h}] + \mathbb{E}[X_{1,t+h} X_{2,t}] \\
 &= \mathbb{E}[X_{1,t} X_{1,t+h}] + \mathbb{E}[X_{2,t} X_{2,t+h}] + \mathbb{E}[X_{1,t}] \mathbb{E}[X_{2,t+h}] + \mathbb{E}[X_{1,t+h}] \mathbb{E}[X_{2,t}] \\
 &= \gamma_{X_1}(h) + \gamma_{X_2}(h) \quad \text{other terms are zero}
 \end{aligned}$$

and so the process is certainly stationary, as this ACVF does not depend on t . We have that

$$Y_t = (1 - \phi B)^{-1} Z_{1,t} + (1 - \varphi B)^{-1} Z_{2,t}$$

so that

$$(1 - \phi B)(1 - \varphi B)Y_t = (1 - \varphi B)Z_{1,t} + (1 - \phi B)Z_{2,t} = (Z_{1,t} + Z_{2,t}) - \varphi Z_{1,t-1} - \phi Z_{2,t-1}.$$

The process

$$W_t = (Z_{1,t} + Z_{2,t}) - \varphi Z_{1,t-1} - \phi Z_{2,t-1}$$

on the right hand side is a stationary process; the mean of this process is zero, and the ACVF of this process is the sum of the two ACVFs of the two independent MA components. If $\phi = \varphi$, then $\{W_t\}$ is an MA(1) process with parameter

$$\theta = -(\phi + \varphi)$$

and driving white noise process $\{Z_t\} \sim \text{WN}(0, 2)$. If $\phi \neq \varphi$, then the ACVF is equal to

$$\begin{aligned}
 h = 0 \quad & (1 + \varphi^2) + (1 + \phi^2) = 2 + \phi^2 + \varphi^2 = \sigma_Z^2(1 + \theta^2) \\
 h = 1 \quad & -(\phi + \varphi) = \sigma_Z^2 \theta
 \end{aligned}$$

say, and is zero for all other h . This is the ACVF of an MA(1) process with parameter θ . The ACF at $h = 1$ is

$$\rho_1 = -\frac{\phi + \varphi}{2 + \phi^2 + \varphi^2} = \frac{\theta}{1 + \theta^2}.$$

say, where

$$(1 + \theta^2)\rho_1 = \theta$$

and so

$$\theta = \frac{1 \pm \sqrt{1 - 4\rho_1^2}}{2}$$

say; note that

$$\rho_1^2 = \frac{(\phi + \varphi)^2}{(2 + \phi^2 + \varphi^2)^2} < \frac{1}{4}.$$

Therefore the $\{W_t\}$ is equivalent to an MA(1) process with parameter θ , where the driving white noise process is

$$\{Z_t\} \sim \text{WN}(0, \sigma_Z^2)$$

where

$$\sigma_Z^2 = \frac{-(\phi + \varphi)}{\theta}.$$

Therefore, $\{Y_t\} \sim \text{ARMA}(2, 1)$; it is stationary and causal as $0 < \phi, \varphi < 1$.

4 MARKS

Computation and Simulation: For illustration take $\phi = 0.9$ and $\varphi = 0.95$,

```
phi<-0.9
varphi<-0.95
r<--(phi+varphi)/(2+phi^2+varphi^2)
th<-(1+c(-1,1))*sqrt(1-4*r^2)/(2*r)
th

## [1] -0.9211095 -1.0856473
```

so that the possible values of θ are

$$\theta = (-0.9211095, -1.0856473)$$

with corresponding σ_Z^2 values

$$\sigma_Z^2 = (2.0084474, 1.7040526).$$

```
library(ltsa)

#Sum of two AR processes
gammaX1<-tacvfARMA(phi=c(phi),maxLag=10)
gammaX2<-tacvfARMA(phi=c(varphi),maxLag=10)
gammaX1+gammaX2

## [1] 15.519568 14.480432 13.519568 12.630432 11.807068 11.044057 10.336462
## [8] 9.679786 9.069929 8.503152 7.976041

phi1<-(phi+varphi)
phi2<--phi*varphi
#ARMA(2,1): First value of theta
sigZ<--(phi+varphi)/th[1]
gammaX<-tacvfARMA(phi=c(phi1,phi2),theta=c(-th[1]),sigma2=sigZ,maxLag=10)
gammaX

## [1] 15.519568 14.480432 13.519568 12.630432 11.807068 11.044057 10.336462
## [8] 9.679786 9.069929 8.503152 7.976041

#ARMA(2,1): Second value of theta
sigZ<--(phi+varphi)/th[2]
gammaX<-tacvfARMA(phi=c(phi1,phi2),theta=c(-th[2]),sigma2=sigZ,maxLag=10)
gammaX

## [1] 15.519568 14.480432 13.519568 12.630432 11.807068 11.044057 10.336462
## [8] 9.679786 9.069929 8.503152 7.976041

#ACF
(gammaX1+gammaX2)/(gammaX1[1]+gammaX2[1])

## [1] 1.0000000 0.9330435 0.8711304 0.8138391 0.7607859 0.7116214 0.6660277
## [8] 0.6237149 0.5844189 0.5478987 0.5139345

#First value of theta
gammaX<-ARMAacf(ar=c(phi1,phi2),ma=c(th[1]),lag.max=10)
gammaX

## 0 1 2 3 4 5 6 7
## 1.0000000 0.9330435 0.8711304 0.8138391 0.7607859 0.7116214 0.6660277 0.6237149
## 8 9 10
## 0.5844189 0.5478987 0.5139345
```

```
#Second value of theta
gammaX<-ARMAacf(ar=c(phi1,phi2),ma=c(th[2]),lag.max=10)
gammaX

##           0           1           2           3           4           5           6           7
## 1.0000000 0.9330435 0.8711304 0.8138391 0.7607859 0.7116214 0.6660277 0.6237149
##           8           9          10
## 0.5844189 0.5478987 0.5139345
```

Simulation:

```
n<-100000
X1<-arima.sim(n,model=list(ar=phi))
X2<-arima.sim(n,model=list(ar=varphi))
Y<-X1+X2
acf(Y,plot=FALSE,lag.max=10)

##
## Autocorrelations of series 'Y', by lag
##
##           0           1           2           3           4           5           6           7           8           9          10
## 1.000 0.933 0.872 0.815 0.762 0.714 0.668 0.627 0.588 0.552 0.518

sigZ<--(phi+varphi)/th[1]
Y<-arima.sim(n,model=list(ar=c(phi1,phi2),ma=th[1]),sd=sqrt(sigZ))
acf(Y,plot=FALSE,lag.max=10)

##
## Autocorrelations of series 'Y', by lag
##
##           0           1           2           3           4           5           6           7           8           9          10
## 1.000 0.933 0.871 0.813 0.760 0.711 0.666 0.623 0.584 0.547 0.513

sigZ<--(phi+varphi)/th[2]
Y<-arima.sim(n,model=list(ar=c(phi1,phi2),ma=th[2]),sd=sqrt(sigZ))
acf(Y,plot=FALSE,lag.max=10)

##
## Autocorrelations of series 'Y', by lag
##
##           0           1           2           3           4           5           6           7           8           9          10
## 1.000 0.935 0.875 0.819 0.768 0.720 0.676 0.635 0.597 0.561 0.528
```