

MATH 545 - ASSIGNMENT 1

Please upload a single pdf with your solutions to myCourses by 11.59pm on Sunday 26th January.

1. For each of the following time series processes, $\{X_t\}$, derive the ACVF, $\gamma_X(h)$, $h = 0, \pm 1, \pm 2, \dots$ from the model equation. In each case, $\{Z_t\} \sim \text{WN}(0, 1)$.

- (a) $X_t = Z_t - Z_{t-1}$. 3 MARKS
 (b) $X_t = Z_t - Z_{t-2}$. 3 MARKS
 (c) $X_t = \nabla^2 Z_t = (1 - B)^2 Z_t$. 3 MARKS
 (d) $X_t = \lambda^2 X_{t-2} + Z_t$, for constant λ where $-1 < \lambda < 1$. 3 MARKS

Assume that $\{X_t\}$ is stationary in all cases. Show your working.

Solution: Note that all of the processes have mean zero by stationarity. The ACVF is an even function, so we evaluate only for $h \geq 0$.

- (a) $X_t = Z_t - Z_{t-1}$. This is the MA(1) process with parameter $\theta_1 = -1$. Therefore

$$\mathbb{E}[X_t X_{t+h}] = \mathbb{E}[(Z_t - Z_{t-1})(Z_{t+h} - Z_{t+h-1})] = \mathbb{E}[Z_t Z_{t+h}] - \mathbb{E}[Z_t Z_{t+h-1}] - \mathbb{E}[Z_{t-1} Z_{t+h}] + \mathbb{E}[Z_{t-1} Z_{t+h-1}]$$

Evaluating for $h = 0, 1, 2, \dots$ and noting that $\mathbb{E}[Z_t Z_s] = 0$ if $t \neq s$, we have

$$\gamma_X(h) = \mathbb{E}[X_t X_{t+h}] = \begin{cases} 2 & h = 0 & \text{first and fourth terms non-zero} \\ -1 & h = 1 & \text{second term non-zero} \\ 0 & h \geq 2 \end{cases}$$

3 MARKS

```
#Check using R function
library(ltsa)
res<-rbind(0:5,tacvfARMA(theta=1,maxLag=5))
rownames(res)<-c('h','AVCF'); colnames(res)<-rep(' ',6); res

##
## h      0  1  2  3  4  5
## AVCF  2 -1  0  0  0  0
```

- (b) $X_t = Z_t - Z_{t-2}$. This is an MA(2) process with parameter $\theta_1 = 0$ and $\theta_2 = -1$. Therefore

$$\mathbb{E}[X_t X_{t+h}] = \mathbb{E}[(Z_t - Z_{t-2})(Z_{t+h} - Z_{t+h-2})] = \mathbb{E}[Z_t Z_{t+h}] - \mathbb{E}[Z_t Z_{t+h-2}] - \mathbb{E}[Z_{t-2} Z_{t+h}] + \mathbb{E}[Z_{t-2} Z_{t+h-2}]$$

Evaluating for $h = 0, 1, 2, \dots$

$$\gamma_X(h) = \mathbb{E}[X_t X_{t+h}] = \begin{cases} 2 & h = 0 & \text{first and fourth terms non-zero} \\ 0 & h = 1 \\ -1 & h = 2 & \text{second term non-zero} \\ 0 & h \geq 3 \end{cases}$$

3 MARKS

```
#Check using R function
res<-rbind(0:5,tacvfARMA(theta=c(0,1),maxLag=5))
rownames(res)<-c('h','AVCF'); colnames(res)<-rep(' ',6); res

##
## h      0  1  2  3  4  5
## AVCF  2  0 -1  0  0  0
```

(c) We have

$$X_t = \nabla^2 Z_t = (1 - B)^2 Z_t = (1 - 2B + B^2)Z_t = Z_t - 2Z_{t-1} + Z_{t-2}.$$

This is an MA(2) process with parameter $\theta_1 = -2$ and $\theta_2 = 1$. Therefore

$$\begin{aligned}\mathbb{E}[X_t X_{t+h}] &= \mathbb{E}[(Z_t - 2Z_{t-1} + Z_{t-2})(Z_{t+h} - 2Z_{t+h-1} + Z_{t+h-2})] \\ &= \mathbb{E}[Z_t Z_{t+h}] - 2\mathbb{E}[Z_t Z_{t+h-1}] + \mathbb{E}[Z_t Z_{t+h-2}] \\ &\quad - 2\mathbb{E}[Z_{t-1} Z_{t+h}] + 4\mathbb{E}[Z_{t-1} Z_{t+h-1}] - 2\mathbb{E}[Z_{t-1} Z_{t+h-2}] \\ &\quad + \mathbb{E}[Z_{t-2} Z_{t+h}] - 2\mathbb{E}[Z_{t-2} Z_{t+h-1}] + \mathbb{E}[Z_{t-2} Z_{t+h-2}]\end{aligned}$$

Evaluating for $h = 0, 1, 2, \dots$

$$\gamma_X(h) = \mathbb{E}[X_t X_{t+h}] = \begin{cases} 6 & h = 0 \quad \text{first, fifth and ninth terms non-zero} \\ -4 & h = 1 \quad \text{second and sixth term non-zero} \\ 1 & h = 2 \quad \text{third term non-zero} \\ 0 & h \geq 3 \end{cases}$$

3 MARKS

```
#Check using R function
res<-rbind(0:5,tacvfARMA(theta=c(2,-1),maxLag=5))
rownames(res)<-c('h','AVCF'); colnames(res)<-rep(' ',6); res

##
## h      0  1  2  3  4  5
## AVCF  6 -4  1  0  0  0
```

(d) If $\lambda = 0$ we have a white noise process, so assume that $\lambda \neq 0$. We have for $h = 0, 1, 2, \dots$

$$\mathbb{E}[X_t X_{t-h}] = \lambda^2 \mathbb{E}[X_{t-2} X_{t-h}] + \mathbb{E}[Z_t X_{t-h}].$$

When $h = 0$, we have

$$\mathbb{E}[Z_t X_t] \equiv \mathbb{E}[Z_t^2] = 1$$

but for $h > 0$, $\mathbb{E}[Z_t X_{t-h}] = 0$. Therefore

$$h = 0 \quad \gamma_X(0) = \lambda^2 \gamma_X(2) + 1$$

$$h = 1 \quad \gamma_X(1) = \lambda^2 \gamma_X(1)$$

$$h = 2 \quad \gamma_X(2) = \lambda^2 \gamma_X(0)$$

$$h = 3 \quad \gamma_X(3) = \lambda^2 \gamma_X(1)$$

The second equation tells that $\gamma_X(1) = 0$, and the fourth equation tells is that $\gamma_X(h) = 0$ for all odd h . Combining first and third equations, we have

$$\gamma_X(0) = \lambda^2 \gamma_X(2) + 1 = \lambda^4 \gamma_X(0) + 1 \quad \implies \quad \gamma_X(0) = \frac{1}{1 - \lambda^4}$$

and hence for $h \geq 2$ even, from the third equation, we have

$$\gamma_X(h) = \frac{1}{1 - \lambda^4} \lambda^h.$$

This is equivalent to an AR(1) model with $\phi = \lambda^2$ on the even-numbered time-points.

```
lamsq<-0.5
res<-rbind(0:5,tacvfARMA(phi=c(lamsq),maxLag=5)) #AR(1) model on even t.
rownames(res)<-c('h','AVCF'); colnames(res)<-rep(' ',6); res
```

```
##
## h      0.000000 1.000000 2.000000 3.000000 4.000000 5.000000
## AVCF 1.333333 0.666667 0.333333 0.166667 0.083333 0.041667

lamsq^c(0:5)/(1-lamsq^2)

## [1] 1.333333 0.666667 0.333333 0.166667 0.083333 0.041667

tacfARMA(phi=c(0,lamsq),maxLag=5) #AR(2) model on each t

## [1] 1.333333 0.000000 0.666667 0.000000 0.333333 0.000000
```

2. For a stationary AR(1) process, we can write the model equation

$$X_t - \phi_1 X_{t-1} = Z_t \quad t \in \mathbb{Z}$$

or equivalently using the backshift operator

$$(1 - \phi_1 B)X_t = Z_t$$

where we assume $|\phi_1| < 1$. Using the inverse operator expression $(1 - \phi_1 B)^{-1}$, and the identity

$$(1 - \phi_1 B)^{-1}(1 - \phi_1 B)X_t = X_t \quad (1)$$

verify that the inverse operator can be expressed as

$$(1 - \phi_1 B)^{-1} = \sum_{j=0}^{\infty} \phi_1^j B^j \quad (2)$$

and hence that we may write

$$X_t = \sum_{j=0}^{\infty} \phi_1^j Z_{t-j}$$

as given by the recursion from lectures. Specifically, verify that identity (1) holds for the expansion claimed in (2).

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Solution: We need to verify

$$(1 - \phi_1 B)^{-1}(1 - \phi_1 B)X_t = X_t$$

or equivalently in terms of the corresponding polynomials

$$(1 - \phi_1 z)^{-1}(1 - \phi_1 z) = 1.$$

Expanding using the geometric progression, we have for the left-hand side

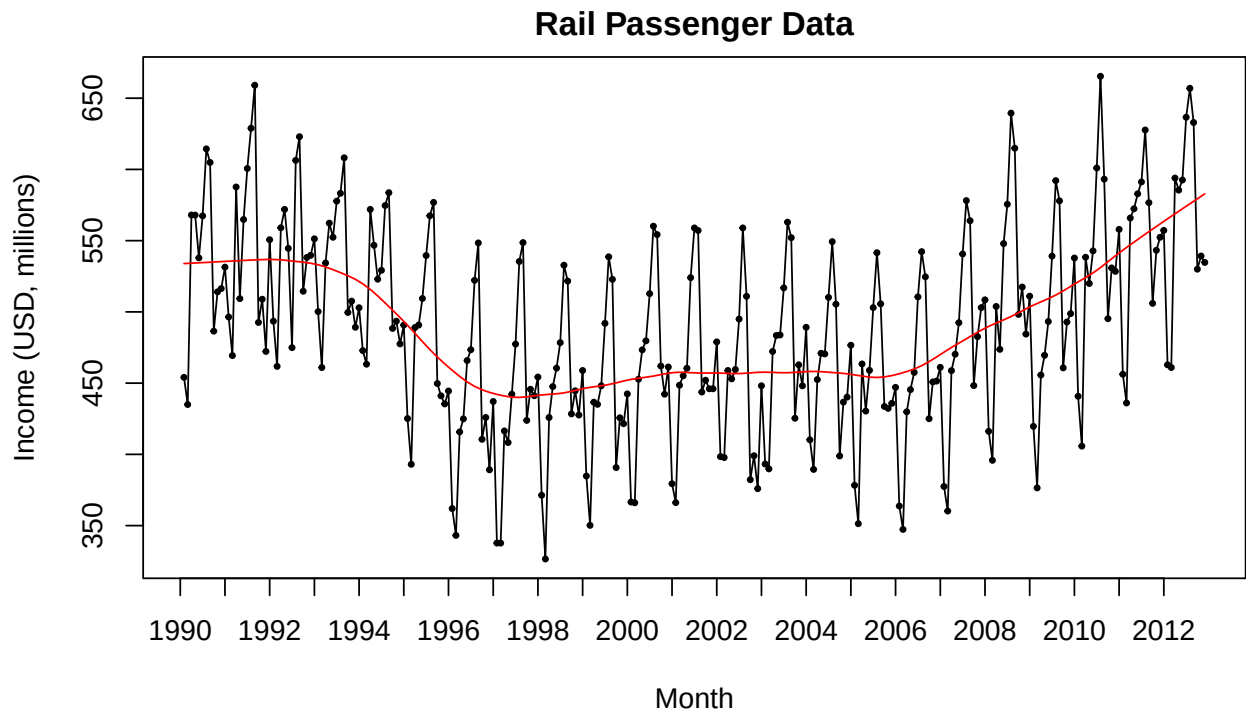
$$\begin{aligned} (1 - \phi_1 z)^{-1}(1 - \phi_1 z) &= (1 + \phi_1 z + \phi_1^2 z^2 + \cdots)(1 - \phi_1 z) \\ &= 1 + (\phi_1 - \phi_1)z + (\phi_1^2 - \phi_1^2)z^2 + \cdots = 1. \end{aligned}$$

as required.

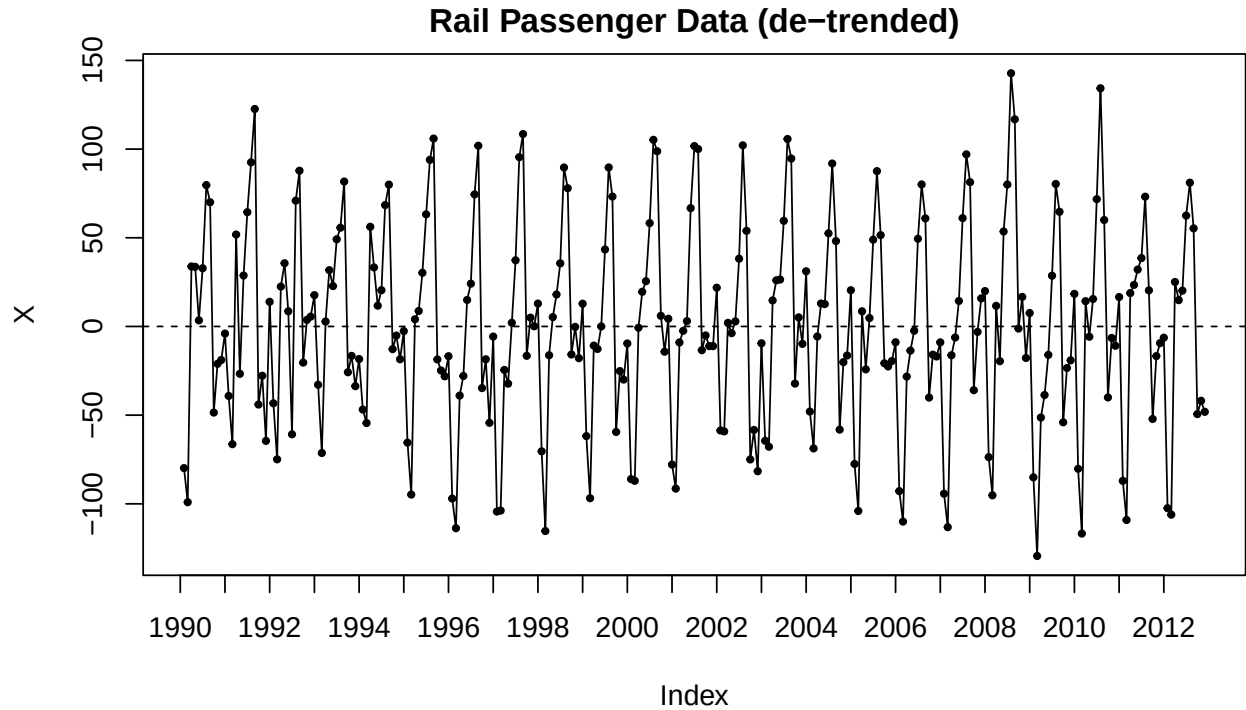
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3. The following 275 data points represent monthly income (in millions of US dollars) from rail passengers on a national rail carrier between 1990 and 2012. In this data set, some pre-processing is necessary to account for external economic factors; for example, the `lowess` smoothing function may be used.

```
file.url<-'https://www.math.mcgill.ca/dstephens/545/Assignments/A1/RailPass.txt'
Rail<-scan(file.url)
par(mar=c(4,4,2,0))
smooth.mu<-lowess(Rail,f=1/5) #Smoothing by local linear fitting
X<-Rail-smooth.mu$y
plot(Rail,pch=19,cex=0.4,xlab='Month',ylab='Income (USD, millions)',xaxt='n')
axis(1,at=c(0:22)*12,seq(1990,2012,by=1))
lines(Rail);title('Rail Passenger Data')
lines(smooth.mu,col='red')
```



```
plot(X, pch=19, cex=0.5, xaxt='n'); lines(X); abline(h=0, lty=2)
title('Rail Passenger Data (de-trended)')
axis(1, at=c(0:22)*12, seq(1990, 2012, by=1))
```

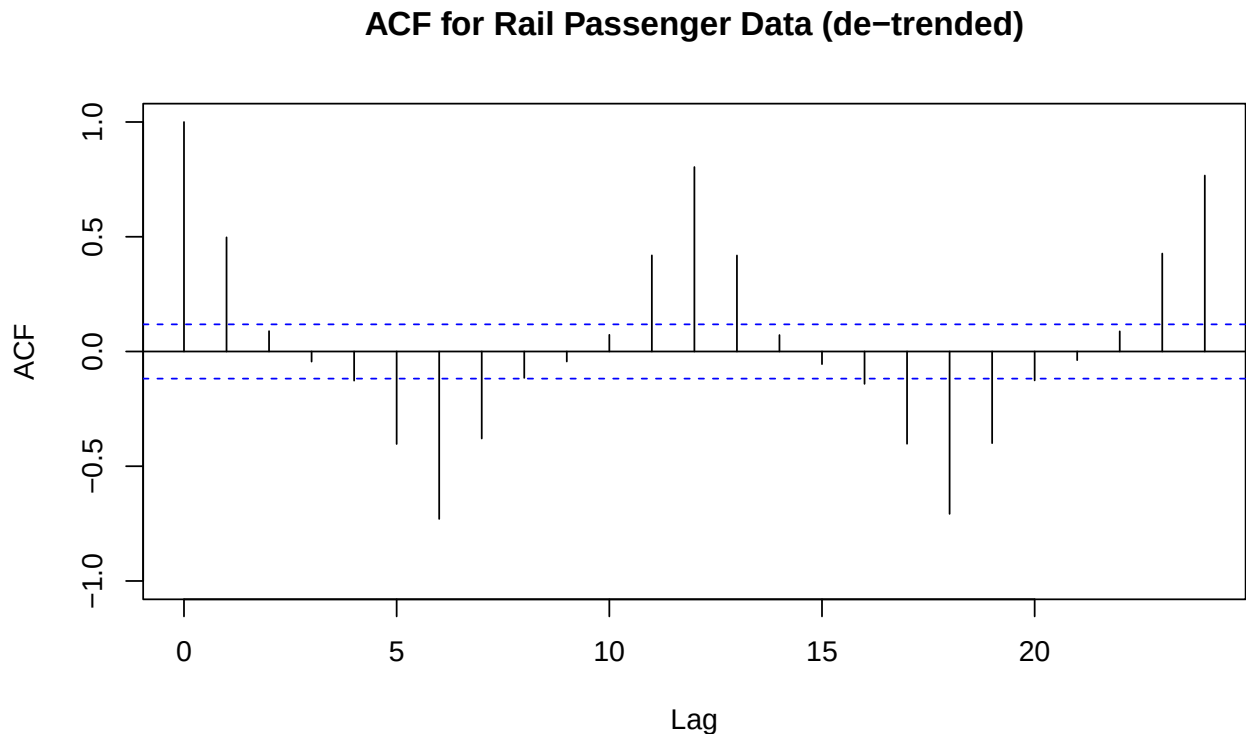


Using software, compute and plot the estimated ACF for the de-trended series to lag 24, and comment on its features.

3 MARKS

Solution: We can compute and plot the ACF using the following code:

```
file.url<-'https://www.math.mcgill.ca/dstephens/545/Assignments/A1/RailPass.txt'
Rail<-scan(file.url)
par(mar=c(4,4,4,0))
smooth.mu<-lowess(Rail,f=1/5) #Smoothing by local linear fitting
X<-Rail-smooth.mu$y
acf(X,lag.max=24,main='ACF for Rail Passenger Data (de-trended)',ylim=range(-1,1))
```



```
acf(X,plot=FALSE)

+
+ Autocorrelations of series 'X', by lag
+
+      0      1      2      3      4      5      6      7      8      9     10
+ 1.000  0.497  0.089 -0.044 -0.127 -0.403 -0.730 -0.379 -0.115 -0.044  0.073
+    11    12    13    14    15    16    17    18    19    20    21
+ 0.419  0.804  0.418  0.072 -0.055 -0.141 -0.402 -0.708 -0.400 -0.126 -0.038
+    22    23    24
+ 0.088  0.427  0.767
```

The ACF has an apparent periodic structure, with strong negative correlation at lag odd multiples of 6, and strong negative correlation at lag even multiples of 6, indicating an annual cycle due to seasonal factors. The ACF magnitude does not appear to be decreasing to zero, indicating non-stationary behaviour.

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