RECURRENCE RELATION METHODS FOR AUTOREGRESSIVE PROCESSES

Given an autoregressive process equation which defines the process $\{X_t\}$ implicitly in terms of $\{Z_t\} \sim \text{WN}(0, \sigma_Z^2)$, we typically need to compute an explicit representation. Suppose that

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t$$

and we want to obtain X_t explicitly in terms of $\{Z_t\}$, say in the form

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

for some constants $\psi_j, j \geq 0$, such that the summation is convergent in mean square (as for the AR(1) process).

We may use the recursion method to compute such a representation: to begin, we substitute in the equivalent expression for X_{t-1}

$$\begin{aligned} X_t &= \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t \\ &= \phi_1(\phi_1 X_{t-2} + \phi_2 X_{t-3} + \cdots + \phi_p X_{t-p-1} + Z_{t-1}) + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t \\ &= \alpha_1^{(2)} X_{t-2} + \alpha_2^{(2)} X_{t-3} + \cdots + \alpha_p^{(2)} X_{t-p-1} + \sum_{j=0}^{1} \psi_j Z_{t-j} \end{aligned} \tag{1}$$

say, where

$$\begin{aligned} \alpha_1^{(2)} &= \phi_1^2 + \phi_2 \\ \alpha_2^{(2)} &= \phi_1\phi_2 + \phi_3 \\ \alpha_3^{(2)} &= \phi_1\phi_3 + \phi_4 \\ &\vdots \quad \vdots \\ \alpha_{p-1}^{(2)} &= \phi_1\phi_{p-1} + \phi_p \\ \alpha_p^{(2)} &= \phi_1\phi_p \end{aligned}$$

and $\psi_1 = 0, \psi_2 = \phi_1$. Let $\alpha^{(2)} = (\alpha_1^{(2)}, \alpha_2^{(2)}, \ldots, \alpha_p^{(2)})^\top$ be the vector of coefficients. Notice that, if we define the vector $\alpha^{(1)} = (\phi_1, \phi_2, \ldots, \phi_p)^\top$, we can see that in fact

$$\begin{aligned} \alpha_1^{(2)} &= \phi_1\alpha_1^{(1)} + \alpha_2^{(1)} \\ \alpha_2^{(2)} &= \phi_2\alpha_1^{(1)} + \alpha_3^{(1)} \\ \alpha_3^{(2)} &= \phi_3\alpha_1^{(1)} + \alpha_4^{(1)} \end{aligned}$$

and so on, until finally

$$\alpha_p^{(2)} = \phi_p\alpha_1^{(1)}.$$

Therefore we may write

$$\alpha^{(2)} = \mathbf{A}\alpha^{(1)}$$

for $(p \times p)$ matrix $\mathbf{A}$, as the relationship between the vectors of coefficients is linear. In fact, we have that

$$\mathbf{A} = \begin{bmatrix} \phi_1 & 1 & 0 & 0 & \cdots & 0 \\ \phi_2 & 0 & 1 & 0 & \cdots & 0 \\ \phi_3 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \phi_{p-1} & 0 & 0 & 0 & \ddots & 1 \\ \phi_p & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

We then proceed by applying the same recursive calculation as in (3) to obtain

$$X_t = \alpha_1^{(3)} X_{t-3} + \alpha_2^{(3)} X_{t-4} + \cdots + \alpha_p^{(3)} X_{t-p-2} + \sum_{j=0}^{2} \psi_j Z_{t-j} \tag{2}$$

where we have an identical recursive relationship

$$\begin{aligned}
\alpha_1^{(3)} &= \phi_1 \alpha_1^{(2)} + \alpha_2^{(2)} \\
\alpha_2^{(3)} &= \phi_2 \alpha_1^{(2)} + \alpha_3^{(2)} \\
\alpha_3^{(3)} &= \phi_3 \alpha_1^{(2)} + \alpha_4^{(2)} \\
&\vdots \quad \vdots \\
\alpha_{p-1}^{(3)} &= \phi_{p-1} \alpha_1^{(2)} + \alpha_p^{(2)} \\
\alpha_p^{(3)} &= \phi_p \alpha_1^{(2)}
\end{aligned}$$

and with $\psi_2 = \alpha_1^{(2)}$. Therefore, we have a recursive relationship

$$\alpha^{(3)} = \mathbf{A}\alpha^{(2)} = \mathbf{A}^2 \alpha^{(1)}$$

that is given by the linear recurrence relation for $r \geq 2$

$$\alpha^{(r)} = \mathbf{A}\alpha^{(r-1)}$$

with $\psi_j = \alpha_1^{(j)}$. By repeatedly applying the recursion, we obtain

$$\alpha^{(r)} = \mathbf{A}\alpha^{(r-1)} = \mathbf{A}^2 \alpha^{(r-2)} = \cdots = \mathbf{A}^{r-1} \alpha^{(1)}.$$

which defines the coefficients in the representation

$$X_t = \alpha_1^{(r)} X_{t-r} + \alpha_2^{(r)} X_{t-r-1} + \cdots + \alpha_p^{(r)} X_{t-p-r} + \sum_{j=0}^{r-1} \psi_j Z_{t-j} \tag{3}$$

Now, as in the AR(1) case, in order for the representation

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

to hold, we need the dependence on the $X_{t-r}, X_{t-r-1}, \ldots, X_{t-r-p}$ terms to vanish as $r \longrightarrow \infty$. To establish this, we examine the recursion for the coefficients, and recognize that the components of the vector $\alpha^{(r)}$ converge to zero if and only if the matrix $\mathbf{A}^r$ converges to the zero matrix as $r \longrightarrow \infty$. We can check this by diagonalizing $\mathbf{A}$, that is, writing

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$$

where $\mathbf{D}$ a diagonal matrix with diagonal elements equal to the eigenvalues of $\mathbf{A}$, and $\mathbf{V}$ is the matrix whose columns are the corresponding eigenvectors. If all the eigenvalues are less than 1 in absolute value, then we have that

$$\mathbf{A}^r = \mathbf{V}\mathbf{D}^r\mathbf{V}^{-1} \longrightarrow \mathbf{0}_{p,p}.$$

Once we have computed

$$\alpha^{(r)} = \mathbf{A}^{r-1}\alpha^{(1)}$$

for each r, we can generate the ψ_j values by recalling that $\psi_j = \alpha_1^{(j)}$.

Example: For $p = 3$, consider the process

$$X_t = X_{t-1} - 0.19X_{t-2} - 0.03X_{t-3} + Z_t$$

so that $(\phi_1, \phi_2, \phi_3) = (1, -0.19, -0.03)$. Here the matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} 1.00 & 1 & 0 \\ -0.19 & 0 & 1 \\ -0.03 & 0 & 0 \end{bmatrix}$$

We can check the eigen decomposition in R

```
A<-matrix(c(1,-0.19,-0.03,1,0,0,0,1,0),3,3)
eigen(A)

+ eigen() decomposition
+ $values
+ [1]  0.6  0.5 -0.1
+
+ $vectors
+             [,1]        [,2]       [,3]
+ [1,] -0.92747779 -0.89314199  0.6593805
+ [2,]  0.37099112  0.44657100 -0.7253185
+ [3,]  0.04637389  0.05358852  0.1978141
```

indicating that the eigen values are 0.6, 0.5 and -0.1. Thus, $\mathbf{A}^r$ should converge to the zero matrix as $r \longrightarrow \infty$; we can check that also

```
A.r<-diag(c(1,1,1))                 #Identity matrix
for(r in 1:100){A.r<-A.r %*% A}     #A.r stores the A^r matrix
A.r

+               [,1]          [,2]          [,3]
+ [1,]  3.359924e-22  5.599874e-22  9.333123e-22
+ [2,] -1.343970e-22 -2.239950e-22 -3.733249e-22
+ [3,] -1.679962e-23 -2.799937e-23 -4.666562e-23
```

This demonstrates that the process can be represented

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

and that this process is well-defined (in mean-square terms). To compute the ψ values, we may use the recursive formula. We perform the calculation out to 1000 steps:

```
nlags<-1000; psi.vec<-numeric(nlags)
psi0<-1                                     #psi_0 = 1
psi.vec[1]<-1                               #psi_1 = 1
alpha.mat<-matrix(0,ncol=nlags,nrow=3)
alpha.mat[,1]<-c(1,-0.19,-0.03)              #initialize the recursion
for(r in 2:nlags){
    alpha.mat[,r]<-A %*% alpha.mat[,r-1]    #apply the recursive formula
    psi.vec[r]<-alpha.mat[1,r]              #psi_r, r > 1
}
psi.vec[1:12]                               #The first 12 psi values starting at psi_1

+  [1] 1.00000000 0.81000000 0.59000000 0.40610000 0.26970000 0.17484100
+  [7] 0.11141500 0.07010421 0.04369013 0.02702788 0.01662363 0.01017763

ARMAtoMA(ar = c(1,-0.19,-0.03),lag.max=12)    #Check
```

```
+  [1] 1.00000000 0.81000000 0.59000000 0.40610000 0.26970000 0.17484100
+  [7] 0.11141500 0.07010421 0.04369013 0.02702788 0.01662363 0.01017763

psi.vec[nlags]                                  #psi_{999} value

+ [1] 7.285424e-222
```

Once we have the representation in terms of ψ values, we may compute from first principles the autocovariances. Recall that for the zero mean process, by standard calculations

$$\text{Cov}[X_t, X_{t+h}] = \mathbb{E}[X_t X_{t+h}] = \mathbb{E}\left[\left(\sum_{j=0}^{\infty}\psi_j Z_{t-j}\right)\left(\sum_{k=0}^{\infty}\psi_k Z_{t-k}\right)\right] = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\psi_j\psi_k\mathbb{E}[Z_{t-j}Z_{t+h-k}]$$

$$= \sigma_Z^2\sum_{j=0}^{\infty}\psi_j\psi_{j+h}$$

where, as usual $\mathbb{E}[Z_{t-j}Z_{t+h-k}] = \sigma_Z^2$ if $k = j+h$, and is zero otherwise. Truncating the infinite sum by assuming that $\psi_j = 0$ for $j \geq 1000$ yields a computational formula for the covariances.

```
acvs<-numeric(nlags)
psi0.vec<-c(psi0,psi.vec)
acvs[1]<-sum(psi0.vec^2)                        #h=0
for(i in 2:nlags){
    acvs[i]<-sum(psi0.vec[1:(nlags-i+1)]*psi0.vec[i:nlags])  #h=i
}
acfs<-acvs/acvs[1]
acvs[1:10]                                  #The first 10 acvs values

+  [1] 3.29283292 2.71643185 2.00930064 1.39439360 0.93113352 0.60591972
+  [7] 0.38717254 0.24411379 0.15237342 0.09437662

acfs[1:10]                                  #The first 10 acfs values

+  [1] 1.00000000 0.82495283 0.61020425 0.42346321 0.28277582 0.18401168
+  [7] 0.11758038 0.07413489 0.04627426 0.02866122
```