MATH 545 - EXERCISES 1
SOLUTIONS

In all cases for a stationary process, we have $\gamma_X(h) = \gamma_X(-h)$ for $h < 0$ as usual.

1. The processes are both zero mean (as we may take expectation on left and right hand side of the definitions, and $\mathbb{E}[Z_t] = 0\ \forall t$))

   (a) This is the zero mean MA(2) model, so the acvf is zero for $h > 2$. In the three other cases of interest: We have

   - $h = 0$:

   $$\gamma_X(0) = \mathbb{E}[X_t^2] = \mathbb{E}[(Z_t + 0.7Z_{t-1} - 0.2Z_{t-2})^2] = (1 + 0.7^2 + (-0.2)^2)\sigma_Z^2 = 1.53\sigma_Z^2$$

   - $h = 1$:

   $$\begin{aligned}\gamma_X(1) = \mathbb{E}[X_t X_{t+1}] &= \mathbb{E}[(Z_t + 0.7Z_{t-1} - 0.2Z_{t-2})(Z_{t+1} + 0.7Z_t - 0.2Z_{t-1})] \\ &= (0.7 + (0.7)(-0.2))\sigma_Z^2 = 0.56\sigma_Z^2\end{aligned}$$

   - $h = 2$:

   $$\begin{aligned}\gamma_X(2) = \mathbb{E}[X_t X_{t+2}] &= \mathbb{E}[(Z_t + 0.7Z_{t-1} - 0.2Z_{t-2})(Z_{t+2} + 0.7Z_{t+1} - 0.2Z_t)] \\ &= -0.20\sigma_Z^2\end{aligned}$$

   (b) This is the MA(q) model with equal coefficients. The acvf is zero for $h > q$, but for $0 \le h \le q$

   $$\begin{aligned}\gamma_X(h) = \mathbb{E}[X_t X_{t+h}] &= \frac{1}{(q+1)^2}\mathbb{E}\left[\left(\sum_{j=0}^{q} Z_{t-j}\right)\left(\sum_{k=0}^{q} Z_{t+h-k}\right]\right) \\ &= \frac{1}{(q+1)^2}\sum_{j=0}^{q}\sum_{k=0}^{q}\mathbb{E}\left[Z_{t-j}Z_{t+h-k}\right] \\ &= \frac{1}{(q+1)^2}\sum_{j=0}^{q-h}\mathbb{E}\left[Z_{t-j}^2\right] = \frac{\sigma_Z^2(q-h+1)}{(q+1)^2}\end{aligned}$$

   as in line 2, only the term with $k = j + h$ is retained, and we also need $0 \le j + h \le q$, yielding line 3.

2. Noting from the second equation that $X_t - \phi X_{t-1} = Y_t$, so that $(1 - \phi\mathrm{B})X_t = Y_t$, we have

   $$(1 - \phi\mathrm{B})X_t = Y_t = (1 + \theta\mathrm{B})Z_t$$

   (this is termed an ARMA$(1,1)$ process, which is a combination of AR(1) and MA(1) processes). We seek the linear process form

   $$X_t = \sum_{j=0}^{\infty}\psi_j Z_{t-j} \qquad \text{such that} \quad \sum_{j=0}^{\infty}|\psi_j| < \infty.$$

   in which case must have

   $$(1 - \phi\mathrm{B})\left(\sum_{j=0}^{\infty}\psi_j\mathrm{B}^j\right)Z_t = (1 + \theta\mathrm{B})Z_t$$

Equivalently, from results on the AR(1) process, we have the convergent expansion

$$(1-\phi \mathrm{B})^{-1} = 1 + \phi \mathrm{B} + \phi^2 \mathrm{B}^2 + \cdots$$

and so, by the usual calculation

$$X_t = (1-\phi \mathrm{B})^{-1}(1+\theta \mathrm{B})Z_t = (1 + \phi \mathrm{B} + \phi^2 \mathrm{B}^2 + \cdots)(1+\theta \mathrm{B})Z_t$$

from which we can identify the coefficients directly. Equating coefficients in the previous expression yields

$$(1-\phi \mathrm{B})(\psi_0 + \psi_1 \mathrm{B} + \psi_2 \mathrm{B}^2 + \cdots) = (1+\theta \mathrm{B})$$

and hence

$$\begin{array}{rcll} 1 & : & \psi_0 = 1 & \\ \mathrm{B} & : & -\psi_0\phi + \psi_1 = \theta & \Longrightarrow \quad \psi_1 = \theta + \phi \\ \mathrm{B}^2 & : & -\psi_1\phi + \psi_2 = 0 & \Longrightarrow \quad \psi_2 = \phi\psi_1 = \phi(\theta+\phi) \\ \mathrm{B}^3 & : & -\psi_2\phi + \psi_3 = 0 & \Longrightarrow \quad \psi_3 = \phi\psi_2 = \phi^2(\theta+\phi) \\ \vdots & & \vdots & \\ \mathrm{B}^j & : & -\psi_{j-1}\phi + \psi_j = 0 & \Longrightarrow \quad \psi_j = \phi\psi_{j-1} = \phi^{j-1}(\theta+\phi) \\ \vdots & & \vdots & \end{array}$$

Thus

$$\sum_{j=0}^{\infty} |\psi_j| = 1 + |\theta+\phi| + |\theta+\phi| \sum_{j=2}^{\infty} |\phi|^{j-1} < \infty$$

as $|\phi| < 1$, and hence $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$ is mean-square convergent, and hence $\{X_t\}$ is well-defined and stationary. For the acvf, for $h = 0$,

$$\gamma_X(0) = \sigma_Z^2 \sum_{j=0}^{\infty} \psi_j^2 = \sigma_Z^2 \left[1 + (\theta+\phi)^2 \sum_{j=1}^{\infty} \{\phi^{j-1}\}^2 \right] = \sigma_Z^2 \left[1 + \frac{(\theta+\phi)^2}{1-\phi^2} \right]$$

and for $h \geq 1$,

$$\begin{aligned} \gamma_X(h) &= \sigma_Z^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} = \sigma_Z^2 \left[\phi^{h-1}(\theta+\phi) + (\theta+\phi)^2 \phi^h \sum_{j=1}^{\infty} \{\phi^{j-1}\}^2 \right] \\ &= \sigma_Z^2 \phi^{h-1}(\theta+\phi) \left[1 + \frac{\phi(\theta+\phi)}{1-\phi^2} \right]. \end{aligned}$$

3. By direct calculation, the RHS is zero mean, so $\mathbb{E}[X_t] = 0$ for all t. However

$$\mathrm{Var}[X_t] = \mathbb{E}[X_t^2] = \mathbb{E}\left[\left(Z_t + \lambda \sum_{j=1}^{\infty} Z_{t-j} \right)^2 \right] = \mathbb{E}[Z_t^2] + \lambda^2 \sum_{j=1}^{\infty} \mathbb{E}[Z_{t-j}^2]$$

as $\{Z_t\}$ is an uncorrelated series. The sum is divergent, so $\mathrm{Var}[X_t]$ is not finite, and hence $\{X_t\}$ is not stationary. However,

$$Y_t = X_t - X_{t-1} = \left(Z_t + \lambda \sum_{j=1}^{\infty} Z_{t-j} \right) - \left(Z_{t-1} + \lambda \sum_{j=1}^{\infty} Z_{t-1-j} \right) = Z_t + (\lambda - 1) Z_{t-1}$$

so $\{Y_t\} \sim \mathrm{MA}(1)$ with parameter $\theta = (\lambda - 1)$.

4. As $\{X_t\} \sim$ AR(1), we have for $\Phi(\mathrm{B}) = (1 - \phi\mathrm{B})$ that

$$X_t = \{\Phi(\mathrm{B})\}^{-1} Z_t$$

for some white noise process $\{Z_t\} \sim \mathrm{WN}(0, \sigma_Z^2)$. Now, if $\gamma_X(h) = \phi^{|h|}$ for each h, we have that $\gamma_X(0) = \sigma_Z^2/(1-\phi^2) = 1$, which defines the value of σ_Z^2 for any fixed ϕ, or vice-versa. Using the backshift notation, we have

$$Y_t = (1 - \mathrm{B})X_t = (1 - \mathrm{B})\{\Phi(\mathrm{B})\}^{-1} Z_t = \Psi(\mathrm{B}) Z_t$$

say, where $\Phi(\mathrm{B}) = 1 + \phi\mathrm{B} + \phi^2\mathrm{B}^2 + \cdots$, and hence

$$\Psi(\mathrm{B}) = (1 - \mathrm{B})(1 + \phi\mathrm{B} + \phi^2\mathrm{B}^2 + \cdots) = 1 + (\phi - 1)\mathrm{B} + \phi(\phi - 1)\mathrm{B}^2 + \phi^2(\phi - 1)\mathrm{B}^3 + \cdots = \sum_{j=0}^{\infty} \psi_j \mathrm{B}^j$$

say, with $\psi_0 = 1$, $\psi_j = \phi^{j-1}(\phi - 1)$ for $j \geq 1$. Then we have for $h = 0$

$$\gamma_Y(0) = \sigma_Z^2 \sum_{j=0}^{\infty} \psi_j^2 = \sigma_Z^2 + \sigma_Z^2 \sum_{j=1}^{\infty} \phi^{2(j-1)}(\phi - 1)^2 = \sigma_Z^2 \left(1 + \frac{(1-\phi)^2}{1-\phi^2}\right)$$

and for $h \geq 1$,

$$\begin{aligned}
\gamma_Y(h) = \sigma_Z^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} &= \sigma_Z^2 \phi^{h-1}(\phi - 1) + \sigma_Z^2 \sum_{j=1}^{\infty} \phi^{j-1}(\phi - 1)\phi^{j+h-1}(\phi - 1) \\
&= \sigma_Z^2 \phi^{h-1}(\phi - 1) + \sigma_Z^2 (\phi - 1)^2 \phi^h \sum_{j=1}^{\infty} \phi^{2(j-1)} \\
&= \sigma_Z^2 \phi^{h-1}(\phi - 1) + \sigma_Z^2 \frac{(\phi - 1)^2 \phi^h}{1 - \phi^2} \\
&= \sigma_Z^2 \phi^{h-1}(\phi - 1)\left[1 + \frac{\phi(\phi - 1)}{1 - \phi^2}\right] = \sigma_Z^2 \phi^{h-1}\left(\frac{\phi - 1}{\phi + 1}\right)
\end{aligned}$$

5. Here we can use the factorization of the AR(2) polynomial

$$\Phi(\mathrm{B}) = 1 - \phi_1\mathrm{B} - \phi_2\mathrm{B}^2 = (1 - \alpha_1\mathrm{B})(1 - \alpha_2\mathrm{B})$$

to get the coefficient

$$\psi_j = \sum_{k=0}^{j} \alpha_1^k \alpha_2^{j-k} \qquad j \geq 0$$

Alternate methods include the direct decomposition of the AR(2) used in Assignment 1:

$$\left(\sum_{j=0}^{\infty} \psi_j Z_{t-j}\right) - \phi_1 \left(\sum_{j=0}^{\infty} \psi_j Z_{t-j-1}\right) - \phi_2 \left(\sum_{j=0}^{\infty} \psi_j Z_{t-j-1}\right) = Z_t$$

so that by equating coefficients on LHS and RHS

$$\begin{aligned}
Z_t \quad &: \quad \psi_0 = 1 \\
Z_{t-1} \quad &: \quad \psi_1 - \phi_1\psi_0 = 0 \\
Z_{t-j} \quad &: \quad \psi_j - \phi_1\psi_{j-1} - \phi_2\psi_{j-2} = 0 \quad j \geq 2
\end{aligned}$$

so that $\psi_0 = 1$, $\psi_1 = \phi_1$, $\psi_j = \phi_1\psi_{j-1} + \phi_2\psi_{j-2}, j \geq 2$.

(a) Here we have

$$X_t - 0.9X_{t-1} + 0.2X_{t-2} = (1 - 0.4\mathrm{B})(1 - 0.5\mathrm{B})X_t = (1 - \alpha_1\mathrm{B})(1 - \alpha_2\mathrm{B})X_t = Z_t$$

In this case, as $|\alpha_1|, |\alpha_2| < 1$, we have a stationary, causal process, with linear process expansion that has as the coefficient of Z_{t-j}

$$\begin{array}{rcll} \psi_1 & : & \alpha_1 + \alpha_2 & = & 0.9000 \\ \psi_2 & : & \alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2 & = & 0.6100 \\ \psi_3 & : & \alpha_1^3 + \alpha_1^2\alpha_2 + \alpha_1\alpha_2^2 + \alpha_2^3 & = & 0.3690 \\ \psi_4 & : & \alpha_1^4 + \alpha_1^3\alpha_2 + \alpha_1^2\alpha_2^2 + \alpha_1\alpha_2^3 + \alpha_2^4 & = & 0.2101 \end{array}$$

This can be computed using the `R` command `ARMAtoMA(ar=c(0.9,-0.2),lag.max=4)`

(b) Here we have

$$X_t + X_{t-1} + 0.25X_{t-2} = (1 + 0.5\mathrm{B})(1 + 0.5\mathrm{B})X_t = (1 - \alpha_1\mathrm{B})(1 - \alpha_2\mathrm{B})X_t = Z_t$$

with $\alpha_1 = \alpha_2 = -0.5$, and so again we have a stationary, causal process, with linear process expansion

$$\begin{array}{rcll} \psi_1 & : & \alpha_1 + \alpha_2 & = & -1.0000 \\ \psi_2 & : & \alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2 & = & 0.7500 \\ \psi_3 & : & \alpha_1^3 + \alpha_1^2\alpha_2 + \alpha_1\alpha_2^2 + \alpha_2^3 & = & -0.5000 \\ \psi_4 & : & \alpha_1^4 + \alpha_1^3\alpha_2 + \alpha_1^2\alpha_2^2 + \alpha_1\alpha_2^3 + \alpha_2^4 & = & 0.3125 \end{array}$$

This can be computed using the `R` command `ARMAtoMA(ar=c(-1.0,-0.25),lag.max=4)`

6. (a) $\{X_t\}$ is not invertible with respect to $\{Z_t\}$ as the representation

$$\{(1 - \theta\mathrm{B})\}^{-1} X_t \equiv \Phi(\mathrm{B})X_t = Z_t$$

is *not* convergent in mean-square, as by the standard calculation

$$\{(1 - \theta\mathrm{B})\}^{-1} = 1 + \theta\mathrm{B} + \theta^2\mathrm{B}^2 + \cdots$$

which is a divergent expansion when $\theta > 1$.

(b) Note that we have $Z_t^* = \Psi(\mathrm{B})X_t = \Psi(\mathrm{B})(1 - \theta\mathrm{B})Z_t$, so if we define

$$\Psi(\mathrm{B}) = (1 - \theta^{-1}\mathrm{B})^{-1} = (1 + \theta^{-1}\mathrm{B} + \theta^{-2}\mathrm{B}^2 + \cdots + \theta^{-j}\mathrm{B}^j + \cdots)$$

then

$$\begin{array}{rcl} \Psi^*(\mathrm{B}) = \Psi(\mathrm{B})(1 - \theta\mathrm{B}) & = & (1 - \theta^{-1}\mathrm{B})^{-1} = (1 + \theta^{-1}\mathrm{B} + \theta^{-2}\mathrm{B}^2 + \cdots + \theta^{-j}\mathrm{B}^j + \cdots)(1 - \theta\mathrm{B}) \\ & = & 1 + (\theta^{-1} - \theta)\mathrm{B} + (\theta^{-2} - 1)\mathrm{B}^2 + (\theta^{-2} - \theta^{-1})\mathrm{B}^3 + \cdots \end{array}$$

so that

$$\psi_j^* = \begin{cases} 1 & j = 0 \\ (\theta^{-1} - \theta)\theta^{-(j-1)} & j \geq 1 \end{cases}$$

Thus $Z_t^* = \Psi(\mathrm{B})(1 - \theta\mathrm{B})Z_t$, and hence, for $h > 0$,

$$\begin{array}{rcl} \gamma_{Z^*}(h) & = & \sigma_Z^2 \sum_{j=0}^{\infty} \psi_j^* \psi_{j+h}^* = \sigma_Z^2 \psi_0^* \psi_h^* + \sigma_Z^2 \sum_{j=1}^{\infty} \psi_j^* \psi_{j+h}^* \\ & = & \sigma_Z^2 (\theta^{-1} - \theta)\theta^{-(h-1)} + \sigma_Z^2 (\theta^{-1} - \theta)^2 \theta^{-h} \sum_{j=1}^{\infty} \theta^{-2(j-1)} \\ & = & \sigma_Z^2 (\theta^{-1} - \theta)\theta^{-(h-1)} + \sigma_Z^2 (\theta^{-1} - \theta)^2 \theta^{-h} \frac{1}{1 - \theta^2} = 0. \end{array}$$

so $\{Z_t^*\}$ is a white noise series. For $h = 0$, we have

$$\gamma_{Z^*}(0) = \sigma_Z^2(\theta^{-1} - \theta)^2 \sum_{j=0}^{\infty} \theta^{-2(j-1)} = \sigma_Z^2(\theta^{-1} - \theta)^2 \frac{1}{1-\theta^2} = \frac{\sigma_Z^2}{\theta^2}$$

Now we have

$$X_t = (1 - \theta \mathrm{B})Z_t = (1 - \theta^{-1}\mathrm{B})Z_t^*$$

so $\{X_t\}$ *does* have an invertible representation with respect to $\{Z_t^*\}$.

7. Using the `decompose` function in `R`, which is similar to `stl`, and assuming an additive error model as the data seem to have constant variance, it is straightforward to achieve a decomposition. The code

```
1 Housing<-scan("HousingStarts.txt")
2 Housing.t<-ts(Housing,start=c(1992,1),frequency=12)
3 plot(Housing.t,ylab='Housing starts')
4 points(Housing.t,pch=19,cex=0.5)
5 fit1<-decompose(Housing.t, type = c("additive"))
```

assumes a 12 month cycle (the frequency of the time series object), and reveals a high in June and a low in January.

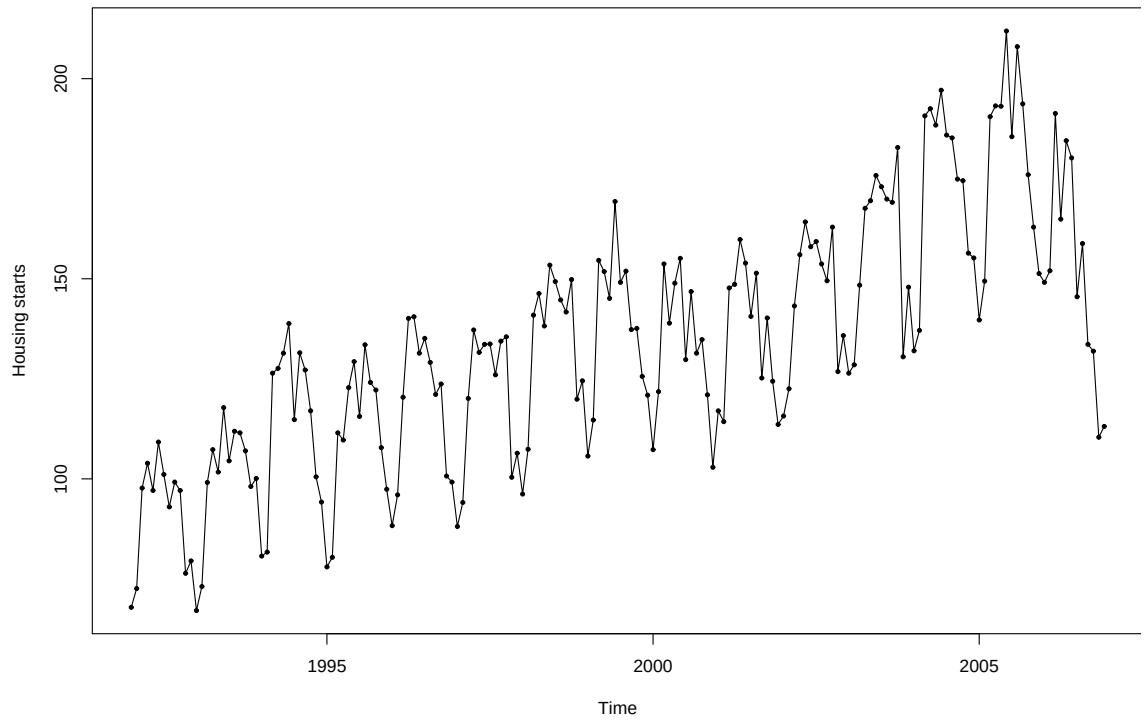

Figure 1: Raw Housing Starts Data

The trend is an almost linear increasing trend until the final segment.

The residual series appears zero mean, and is also not significantly autocorrelated.

The command used here was

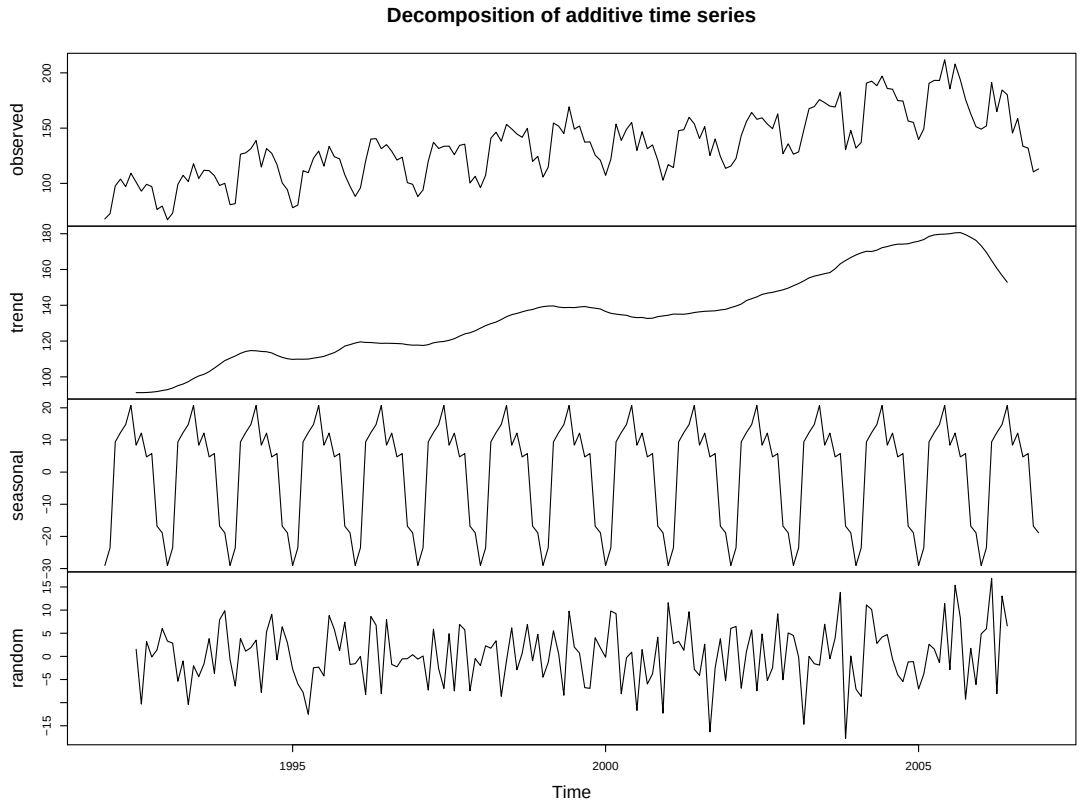

Figure 2: Decomposition of Housing Data

```
1 acf(fit1.res,na.action=na.pass,lag.max=48,
2              main="ACF for Residuals from Additive Model")
```

Thus an adequate model seems to be

$$Y_t = m_t + s_t + Z_t$$

where m_t is the nonparametrically estimated trend, s_t is the annual cycle, and $\{Z_t\}$ is a white noise process with $\sigma_Z^2 \approx 39.79$

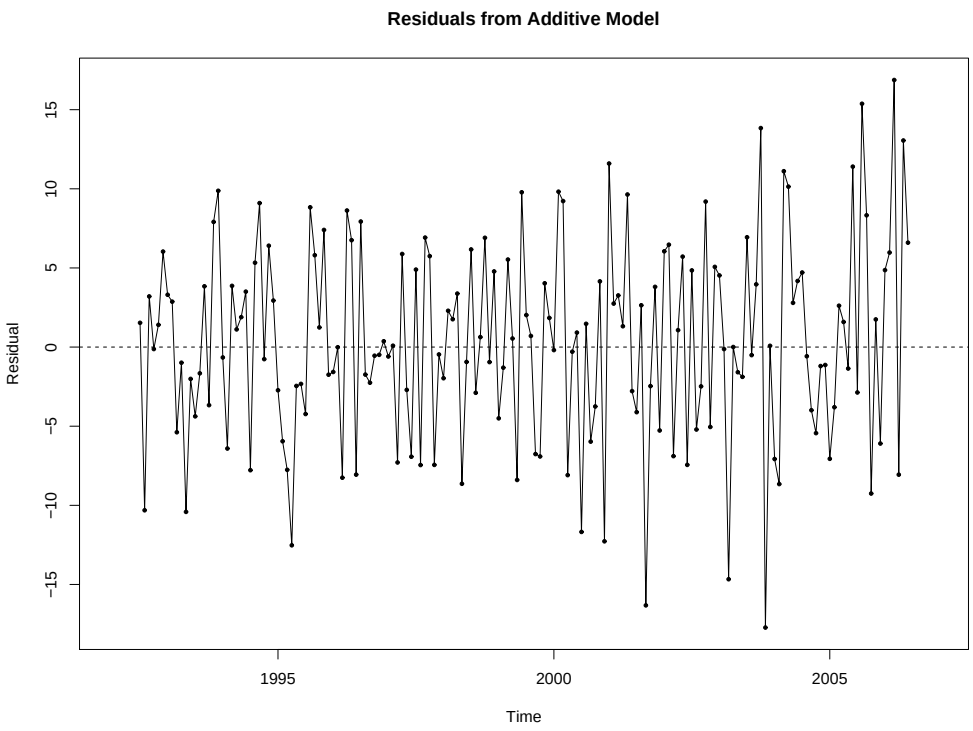

Figure 3: Residuals from Decomposition

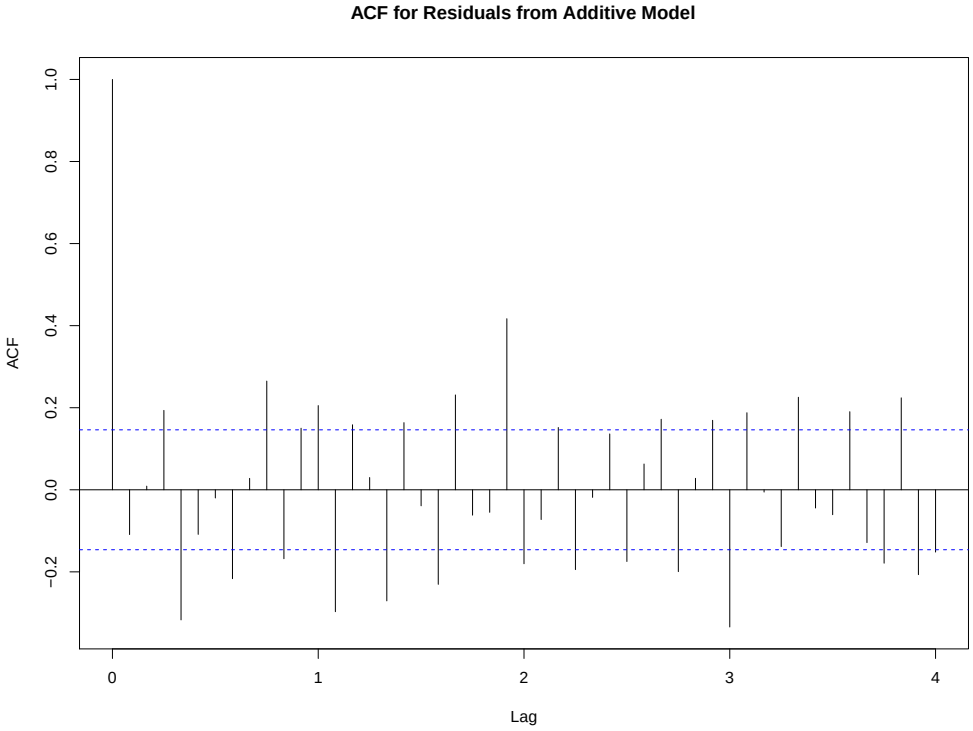

Figure 4: ACF of residuals from decomposition

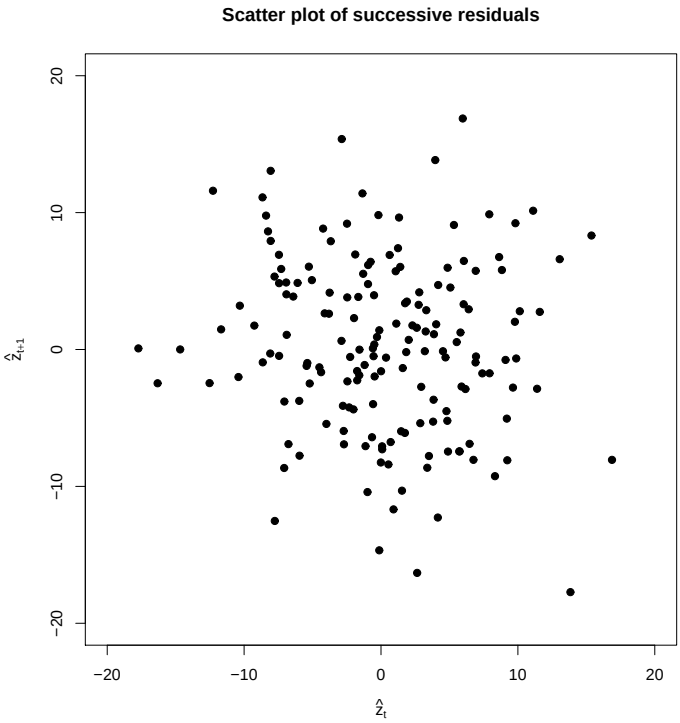

Figure 5: Scatterplot of successive residuals