

MATH 545 - EXERCISES 1

In the following questions, unless otherwise stated, $\{Z_t\} \sim \text{WN}(0, \sigma_Z^2)$.

- Find the autocovariance function (acvf) for the following two processes.

(a) $\{X_t\}$ defined by

$$X_t = Z_t + 0.7Z_{t-1} - 0.2Z_{t-2}$$

(b) $\{X_t\}$ defined by

$$X_t = \frac{1}{q+1} \sum_{j=0}^q Z_{t-j}$$

- Suppose that

$$Y_t = Z_t + \theta Z_{t-1},$$

and that

$$X_t = \phi X_{t-1} + Y_t,$$

for $|\phi|, |\theta| < 1$.

Prove that $\{X_t\}$ is stationary and causal, and derive the autocovariance function for $\{X_t\}$, $\gamma_X(\cdot)$.

- Consider the so-called $\text{MA}(\infty)$ (or causal linear) process $\{X_t\}$ defined by

$$X_t = Z_t + \lambda(Z_{t-1} + Z_{t-2} + \dots)$$

for parameter $\lambda > 0$. Show that $\{X_t\}$ is non-stationary, but that $\{Y_t\}$ defined by

$$Y_t = X_t - X_{t-1}$$

is stationary.

- Suppose that $\{X_t\}$ is stationary with acvf $\gamma_X(h)$, and that $\{Y_t\}$ is defined by

$$Y_t = X_t - X_{t-1}.$$

Find the acvf of $\{Y_t\}$ if $\{X_t\} \sim \text{AR}(1)$, with $\gamma_X(h) = \phi^{|h|}$, $h = 0, \pm 1, \pm 2, \dots$

- For each of the following processes $\{X_t\}$, find the coefficients $\psi_j, j = 1, \dots, 4$ in the causal linear process (or $\text{MA}(\infty)$) representation

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} = \Psi(B)Z_t$$

when such a representation exists; if the representation does not exist, explain why.

(a) $\{X_t\}$ defined implicitly by

$$X_t - 0.9X_{t-1} + 0.2X_{t-2} = Z_t$$

(b) $\{X_t\}$ defined implicitly by

$$X_t + X_{t-1} + 0.25X_{t-2} = Z_t$$

6. An MA(q) process

$$X_t = \Theta(B)Z_t = (1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q)Z_t$$

is termed *invertible* with respect to $\{Z_t\}$ if it can be written in the form $\Phi(B)X_t = Z_t$ where

$$\Phi(B) = \sum_{j=0}^{\infty} \phi_j B^j$$

such that

$$\Phi(B)X_t = \sum_{j=0}^{\infty} \phi_j X_{t-j}$$

is convergent in the mean-square sense.

Suppose that for $\theta > 1$ we have the MA(1) process

$$X_t = Z_t - \theta Z_{t-1}.$$

(a) Show that $\{X_t\}$ is not invertible.

(b) Find a causal linear filter $\Psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$ such that the process $\{Z_t^*\}$ defined by

$$Z_t^* = \Psi(B)X_t$$

is $WN(0, \sigma_{Z^*}^2)$ for some $\sigma_{Z^*}^2$. Hence comment on the property of invertibility for MA(1) processes.

7. The time series contained in the file `HousingStarts.txt` represents 15 years worth of monthly housing start data obtained from the US Bureau of Census. Perform a decomposition analysis for this series, and attempt to characterize the deterministic and stochastic components.

You may use the pre-written functions in R, or the techniques described in lectures.

The file `Housing.R` contains R code for reading in the data.