MATH 523: PROPORTIONAL ODDS MODEL

The proportional odds model is used for ordered multicategory response data; the logistic version of the model is specified for ungrouped data - that is, where $Y \sim Multinomial(1; \pi_1, \ldots, \pi_K)$ as

$$\log\left(\frac{\Pr[Y \leqslant k|x]}{1-\Pr[Y \leqslant k|x]}\right) = \beta_{0k} + x\beta_1 \qquad k = 1, \ldots, K-1$$

for predictor x and parameters β_1. With the logistic link, for two values of the continuous covariate, x_A and x_B,

$$\log\left(\frac{\Pr[Y \leqslant k|x_A]}{1-\Pr[Y \leqslant k|x_A]}\right) - \log\left(\frac{\Pr[Y \leqslant k|x_B]}{1-\Pr[Y \leqslant k|x_B]}\right) = \log\left(\frac{\Pr[Y \leqslant k|x_A]/\Pr[Y > |x_A]}{\Pr[Y \leqslant k|x_B]/\Pr[Y > |x_B]}\right) = (x_A - x_B)\beta_1$$

The quantity

$$\frac{\Pr[Y \leqslant k|x_A]/\Pr[Y > |x_A]}{\Pr[Y \leqslant k|x_B]/\Pr[Y > |x_B]}$$

is termed a *cumulative odds ratio*. For ordered categorical responses, the intercepts β_{0k} should be constrained to be increasing.

Example: Mental Impairment: A study of mental impairment in a sample of 40 adults examined the relationship between the mental impairment outcome (1-Well, 2-Mild, 3-Moderate, 4-Impaired) and two predictors x_1 (LE, number of life events during study period, continuous) and x_2 (SES, socioeconomic status, binary factor).

ID	Outcome Y	SES x_2	LE x_1	ID	Outcome Y	SES x_2	LE x_1
1	Well	1	1	21	Mild	1	9
2	Well	1	9	22	Mild	0	3
3	Well	1	4	23	Mild	1	3
4	Well	1	3	24	Mild	1	1
5	Well	0	2	25	Moderate	0	0
6	Well	1	0	26	Moderate	1	4
7	Well	0	1	27	Moderate	0	3
8	Well	1	3	28	Moderate	0	9
9	Well	1	3	29	Moderate	1	6
10	Well	1	7	30	Moderate	0	4
11	Well	0	1	31	Moderate	0	3
12	Well	0	2	32	Impaired	1	8
13	Mild	1	5	33	Impaired	1	2
14	Mild	0	6	34	Impaired	1	7
15	Mild	1	3	35	Impaired	0	5
16	Mild	0	1	36	Impaired	0	4
17	Mild	1	8	37	Impaired	0	4
18	Mild	1	2	38	Impaired	1	8
19	Mild	0	5	39	Impaired	0	8
20	Mild	1	5	40	Impaired	0	9

The outcome levels are of increasing severity of impairment. These data are available from the course website. The model proposed is

$$\log\left(\frac{\Pr[Y \leqslant k|x]}{1-\Pr[Y \leqslant k|x]}\right) = \beta_{0k} + \beta_1 x_1 + \beta_2 \mathbb{1}_{\{1\}} x_2 \qquad k = 1, 2, 3$$

where the constraint $\beta_{01} < \beta_{02} < \beta_{03}$ respects the ordered nature of the outcomes. The parameter β_2 measures the effect (on the linear predictor scale) of socio-economic status.

The proportional odds model can be fitted with `polr` function from the `MASS` library:

```
>Mental<-read.table("MentalImpairment.txt",header=F)
>Mental1<-Mental[,1:4];>Mental2<-Mental[,5:8];names(Mental2)<-names(Mental1)
>Mental<-rbind.data.frame(Mental1,Mental2)
>names(Mental)<-c("ID","Y","ses","life")
>Mental$ses<-as.factor(Mental$ses)
>Mental$Y<-as.factor(rep(c(1,2,3,4),c(12,12,7,9)))
>summary(polr(Y~life+ses,data=Mental))
Call: polr(formula = Y ~ life + ses, data = Mental)
Coefficients:
       Value Std. Error t value
life  0.3189     0.1210   2.635
ses1 -1.1112     0.6109  -1.819

Intercepts:
    Value   Std. Error t value
1|2 -0.2819  0.6423    -0.4389
2|3  1.2128  0.6607     1.8357
3|4  2.2094  0.7210     3.0644

Residual Deviance: 99.0979
AIC: 109.0979
```



Note: the `polr` function reports parameter estimates for the coefficients for non-intercept terms on the negative scale, that is, it reports $-\widehat{\beta}_1$ and $-\widehat{\beta}_2$. Estimated coefficients with standard errors from lines 5-6 are $\widehat{\beta}_1 = -0.3189\ (0.1210)$, $\widehat{\beta}_2 = 1.1112\ (0.6109)$. The intercepts $\beta_{01} < \beta_{02} < \beta_{03}$ are estimated as -0.2819, 1.2128, 2.2094 (lines 10-12). Using a chi-squared test we can assess the importance of the variables in the model.

```
> drop1(fit.mental.polr,test="Chisq")
Single term deletions

Model: Y ~ life + ses
       Df    AIC    LRT Pr(>Chi)
<none>    109.10
life    1 114.87 7.7765 0.005293 **
ses     1 110.53 3.4292 0.064054 .
---
Signif. codes:  0  ***  0.001  **  0.01  *  0.05  .  0.1     1
```

It is evident that the `ses` predictor is not influential, so this variable can be removed:

```
> summary(polr(Y~life,data=Mental))
Call: polr(formula = Y ~ life, data = Mental)
Coefficients:
      Value Std. Error t value
life 0.2879     0.1175   2.451

Intercepts:
    Value  Std. Error t value
1|2 0.2614 0.5639     0.4636
2|3 1.6563 0.6171     2.6838
3|4 2.5876 0.6938     3.7296

Residual Deviance: 102.5271
AIC: 110.5271
```

The model can also be fitted using the `vglm` function from the `VGAM` library.

```
>library(VGAM)
>fit.mental.vglm<-vglm(Y~life+ses,data=Mental,propodds(reverse=F))
>summary(fit.mental.vglm)
Call: vglm(formula = Y~life + ses, family = propodds, data = Mental)
Pearson Residuals:
                      Min       1Q   Median      3Q    Max
logit(P[Y< = 1]) -1.5687 -0.70508 -0.21026 0.80709 2.7155
logit(P[Y< = 2]) -2.3297 -0.46662  0.26578 0.69049 1.6148
logit(P[Y< = 3]) -3.6874  0.11976  0.20398 0.41914 1.8931

Coefficients:
              Estimate Std. Error  z value
(Intercept):1 -0.28176    0.62304 -0.45223
(Intercept):2  1.21291    0.65119  1.86260
(Intercept):3  2.20947    0.71719  3.08075
life          -0.31888    0.11944 -2.66973
ses1           1.11112    0.61427  1.80884

Number of linear predictors:  3
Names of linear predictors: logit(P[Y< = 1]), logit(P[Y< = 2]), logit(P[Y< = 3])

Dispersion Parameter for cumulative family:   1
Residual deviance: 99.0979 on 115 degrees of freedom
Log-likelihood: -49.54895 on 115 degrees of freedom
```

Using the `propodds` argument without the `reverse=F` option computes via the probabilities

$$\log\left(\frac{\Pr[Y \geqslant k|x]}{1-\Pr[Y \geqslant k|x]}\right) = \beta_{0k} + \beta_1 x_1 + \beta_2 \mathbb{1}_{\{1\}} x_2 \qquad k = 1, 2, 3$$

```
> fit.mental.vglm<-vglm(Y~life+ses,data=Mental,propodds)
> summary(fit.mental.vglm)
Call: vglm(formula = Y ~ life + ses, family = propodds, data = Mental)
Pearson Residuals:
                      Min       1Q   Median       3Q    Max
logit(P[Y> = 2]) -2.7155 -0.80709  0.21026  0.70508 1.5687
logit(P[Y> = 3]) -1.6148 -0.69049 -0.26578  0.46662 2.3297
logit(P[Y> = 4]) -1.8931 -0.41914 -0.20398 -0.11976 3.6874

Coefficients:
              Estimate Std. Error  z value
(Intercept):1  0.28176    0.62304  0.45223
(Intercept):2 -1.21291    0.65119 -1.86260
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life           0.31888    0.11944  2.66973
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Number of linear predictors:  3
Names of linear predictors: logit(P[Y> = 2]), logit(P[Y> = 3]), logit(P[Y> = 4])

Dispersion Parameter for cumulative family:   1
Residual deviance: 99.0979 on 115 degrees of freedom
Log-likelihood: -49.54895 on 115 degrees of freedom
```

The `multinom` function could also be used for these data to fit a conventional multicategory model:

```
>fit.mental.multinom<-multinom(Y ~life + ses, data = Mental)
>summary(it.mental.multinom)

Call: multinom(formula = Y ~ life + ses, data = Mental)
Coefficients:
  (Intercept)      life       ses1
2  -0.6552117 0.2285365 -0.2487293
3  -0.6621855 0.2930040 -1.9437361
4  -1.9025935 0.5601587 -1.6062505

Std. Errors:
  (Intercept)      life      ses1
2   0.8853634 0.1832107 0.9047728
3   0.9582681 0.2199351 1.1081671
4   1.1107458 0.2190388 1.0804573

Residual Deviance: 96.69826
AIC: 114.6983
```

The estimated probabilities arising from the two models are broadly similar, but different:

	Prop. Odds				Multinomial			
ID	1	2	3	4	1	2	3	4
1	0.625	0.256	0.071	0.047	0.602	0.307	0.060	0.032
2	0.115	0.252	0.244	0.389	0.102	0.322	0.105	0.471
3	0.390	0.350	0.145	0.115	0.395	0.399	0.094	0.111
4	0.468	0.329	0.117	0.086	0.467	0.375	0.083	0.075
5	0.285	0.355	0.188	0.172	0.312	0.256	0.289	0.143
6	0.696	0.215	0.054	0.035	0.663	0.268	0.049	0.020
7	0.354	0.356	0.159	0.131	0.384	0.251	0.265	0.100
8	0.468	0.329	0.117	0.086	0.467	0.375	0.083	0.075
9	0.468	0.329	0.117	0.086	0.467	0.375	0.083	0.075
10	0.197	0.326	0.225	0.252	0.196	0.394	0.113	0.297
11	0.354	0.356	0.159	0.131	0.384	0.251	0.265	0.100
12	0.285	0.355	0.188	0.172	0.312	0.256	0.289	0.143
13	0.318	0.357	0.174	0.151	0.324	0.412	0.104	0.160
14	0.100	0.232	0.242	0.426	0.097	0.198	0.289	0.416
15	0.468	0.329	0.117	0.086	0.467	0.375	0.083	0.075
16	0.354	0.356	0.159	0.131	0.384	0.251	0.265	0.100
17	0.152	0.292	0.240	0.317	0.144	0.363	0.111	0.381
18	0.548	0.296	0.092	0.064	0.536	0.343	0.071	0.049
19	0.133	0.273	0.243	0.351	0.137	0.223	0.305	0.336
20	0.318	0.357	0.174	0.151	0.324	0.412	0.104	0.160
21	0.115	0.252	0.244	0.389	0.102	0.322	0.105	0.471
22	0.225	0.339	0.214	0.222	0.245	0.253	0.305	0.197
23	0.468	0.329	0.117	0.086	0.467	0.375	0.083	0.075
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮