MATH 523: EXPONENTIAL FAMILY EXAMPLES

The mass/density function f_Y falls into the Exponential-Dispersion family with scalar parameters (θ, ϕ) if f_Y can be written

$$f_Y(y;\theta,\phi) = \exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right\}$$

for specific functions a, b and c, and the support of the function does not depend on (θ, ϕ).

- θ is the **canonical** parameter
- $b(\theta)$ is the **cumulant function** parameter
- ϕ is the **dispersion** parameter

Note that the function $a(.)$ is often the identity, that is

$$a(\phi) = \phi.$$

If

$$l(\theta,\phi;y) = \log f_Y(y;\theta,\phi)$$

it follows by direct calculations that

$$\mathbb{E}_{f_Y}[\dot{l}_\theta(\theta,\phi;Y)] = 0$$

and

$$\mathbb{E}_{f_Y}\left[\ddot{l}_{\theta\theta}(\theta,\phi;Y)\right] + \mathbb{E}_{f_Y}\left[\left\{\dot{l}_\theta(\theta,\phi;Y)\right\}^2\right] = 0$$

where

$$\dot{l}_\theta(\theta,\phi;y) = \frac{\partial l(\theta,\phi;y)}{\partial\theta} \qquad \ddot{l}_{\theta\theta}(\theta,\phi;y) = \frac{\partial^2 l(\theta,\phi;y)}{\partial\theta^2}$$

Hence in the Exponential Family, we have

$$\mathbb{E}[Y] = \dot{b}(\theta) = \mu \qquad \text{Var}[Y] = \ddot{b}(\theta)a(\phi) = V(\mu)$$

say.

- $\mu = \dot{b}(\theta)$ is the **mean** function
- $V(\mu) = \ddot{b}(\theta)a(\phi)$ is the **variance** function.

POISSON For $y = 0, 1, 2 \ldots$

$$f_Y(y;\lambda) = \frac{\lambda^y e^{-\lambda}}{y!} = \exp\{y\log\lambda - \lambda - \log y!\}$$

- Canonical parameter: $\theta = \log\lambda$
- Cumulant function: $b(\theta) = e^\theta$
- Dispersion parameter: $a(\phi) = 1$
- Mean function: $\mathbb{E}[Y] = \dot{b}(\theta) = e^\theta$
- Variance function: $\text{Var}[Y] = \ddot{b}(\theta)a(\phi) = e^\theta = \mu$, so $V(\mu) = \mu$.

BERNOULLI For $y = 0, 1$

$$f_Y(y;\pi) = \pi^y(1-\pi)^{1-y} = \exp\left\{y\log\left(\frac{\pi}{1-\pi}\right) + \log(1-\pi)\right\}$$

- Canonical parameter: $\theta = \log\left(\frac{\pi}{1-\pi}\right)$
- Cumulant function: $b(\theta) = \log(1+e^\theta)$
- Dispersion parameter: $a(\phi) = 1$
- Mean function: $\mathbb{E}[Y] = \dot{b}(\theta) = \frac{e^\theta}{1+e^\theta} = \pi$
- Variance function: $\text{Var}[Y] = \ddot{b}(\theta)a(\phi) = \frac{e^\theta}{(1+e^\theta)^2} = \mu(1-\mu)$, so $V(\mu) = \mu(1-\mu)$.

BINOMIAL For $z = 0, 1, \ldots, m$

$$f_Z(z;\pi) = \binom{m}{z}\pi^z(1-\pi)^{m-z} = \exp\left\{mz\log\left(\frac{\pi}{1-\pi}\right) + m\log(1-\pi) + \log\binom{m}{z}\right\}$$

Consider the data transformation $y = z/m$. Then for $y = 0, 1/m, 2/m, \ldots, 1$ we have

$$f_Y(y;\pi) = \exp\left\{\frac{y\log\left(\frac{\pi}{1-\pi}\right) + \log(1-\pi)}{1/m} + \log\binom{m}{my}\right\}$$

- Canonical parameter: $\theta = \log\left(\frac{\pi}{1-\pi}\right)$
- Cumulant function: $b(\theta) = \log(1+e^\theta)$
- Dispersion parameter: $a(\phi) = \phi = 1/m$
- Mean function: $\mathbb{E}[Y] = \dot{b}(\theta) = \frac{e^\theta}{1+e^\theta} = \pi$
- Variance function: $\text{Var}[Y] = \ddot{b}(\theta)a(\phi) = \boxed{\frac{e^\theta}{m(1+e^\theta)^2}} = \mu(1-\mu)/m$, so $V(\mu) = \phi\mu(1-\mu)$.



GAMMA For $y > 0$

$$f_Y(y;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}y^{\alpha-1}e^{-\beta y} = \exp\{(\alpha-1)\log y - \beta y + \alpha\log\beta - \log\Gamma(\alpha)\}$$

Consider the reparameterization

$$\nu = \alpha \qquad \mu = \alpha/\beta$$

so that

$$f_Y(y;\mu,\nu) = \frac{1}{\Gamma(\nu)}\left(\frac{\nu y}{\mu}\right)^\nu \exp\left\{-\frac{\nu y}{\mu}\right\}\frac{1}{y}$$

and we have

- Canonical parameter: $\theta = -1/\mu$.
- Cumulant function: $b(\theta) = -\log(-\theta)$
- Dispersion parameter: $a(\phi) = \phi = 1/\nu$.
- Mean function: $\mathbb{E}[Y] = \dot{b}(\theta) = -1/\theta$
- Variance function: $\text{Var}[Y] = \ddot{b}(\theta)a(\phi) = \boxed{(1/(1/\mu^2))(1/\nu)}$, so $V(\mu) = \phi\mu^2$.

