MATH 523: INTERACTIONS

TWO-WAY INTERACTIONS:
An interaction between predictors X_1 and X_2 is present in a model if the effect of changing X_1 on the response Y is **different** for different values of X_2.

(1) **FACTORS:** For two **factor** predictors, an interaction exists if

$$\mathbb{E}[Y|X_1 = x_{11}, X_2 = x_{21}] - \mathbb{E}[Y|X_1 = x_{12}, X_2 = x_{21}]$$
$$\neq \mathbb{E}[Y|X_1 = x_{11}, X_2 = x_{22}] - \mathbb{E}[Y|X_1 = x_{12}, X_2 = x_{22}]$$

for at least one choice of factor levels $(x_{11}, x_{12}, x_{21}, x_{22})$. For example, if X_1 and X_2 have two levels, and the means for the responses for different factor level combinations are

		X_2	
		1	2
X_1	1	0	-3
	2	2	4

then

$$\begin{aligned}\mathbb{E}[Y|X_1 = 1, X_2 = 1] - \mathbb{E}[Y|X_1 = 2, X_2 = 1] &= 0 - 2) = -2\\ \mathbb{E}[Y|X_1 = 1, X_2 = 2] - \mathbb{E}[Y|X_1 = 2, X_2 = 2] &= (-3) - 4 = -7\end{aligned}$$

(2) **COVARIATES**: If

$$\mathbb{E}[Y|X_1 = x_1, X_2 = x_2] = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$$

then

$$\mathbb{E}[Y|X_1 = x_{11}, X_2 = x_2] - \mathbb{E}[Y|X_1 = x_{12}, X_2 = x_2] = \beta_1(x_{11} - x_{12}) + \beta_{12}(x_{11} - x_{12})x_2$$

which is a function of x_2.

(3) **FACTORS and COVARIATES**: Suppose X_1 is a factor taking L levels, and X_2 is a covariate. Then $X_{11} = l$ say defines a subset of the subjects; interaction models allow the modelling of the responses as a function of the covariate to be different for different values of l. For example, the model written

$$1 + X_1 + X_2 + X_1.X_2$$

is a *main effects plus interaction* model for which

$$\mathbb{E}[Y|X_1 = l, X_2 = x_2] = (\beta_0 + \beta_{0l}) + (\beta_1 + \beta_{1l})x_2$$

for $l = 2, \ldots, L$.

HIGHER-ORDER INTERACTIONS

Higher-order interactions represent the changes in lower-order interaction effects at different levels of predictors. For example, for three predictors X_1, X_2, X_3, it is possible to include three **two-way** interactions

$$X_1.X_2 \qquad X_1.X_3 \qquad X_2.X_2$$

but also a **three-way** interaction, $X_1.X_2.X_3$, which represents the changes in the values in each of the two-way interactions at different levels/values of the third variable.

INTERACTIONS WITHOUT MAIN EFFECTS

A commonly adopted convention is to only consider models for which, if a M-way interaction is included, then all m-way interactions between the M variables involved are also included, for $m = 2, 3, \ldots, M-1$. That is, if $M = 4$ and we wish to include a 4-way interaction, we should also include all the

$$\begin{aligned}
\binom{4}{3} &= 4 \quad \text{three-way interactions} \\
\binom{4}{2} &= 6 \quad \text{two-way interactions} \\
\binom{4}{1} &= 4 \quad \text{main effects}
\end{aligned}$$

Such a model is termed a *full factorial* model.

NOTE: Unless there is a specific need or reason to do so, **we do not omit any** lower-order interactions or main effects. That is, for a two-predictor model, we might consider the full factorial model

$$1 + X_1 + X_2 + X_1.X_2$$

but **not** the model

$$1 + X_1 + X_1.X_2$$

which omits the X_2 main effect. The implication of the second model is that X_2 is influential in explaining the response, but only for **specific** values of X_1. For example, for two factor predictors with J and K levels, the second model might imply

$$\mathbb{E}[Y|X_1 = j, X_2 = k] = \begin{cases} \beta_0 & j = 1, k = 1, \ldots, K \\ \beta_0 + \beta_{1j} & j = 2, \ldots, J, k = 1 \\ \beta_0 + \beta_{1j} + \gamma_{jk} & j = 2, \ldots, J, k = 1, \ldots, K \end{cases}$$

so that at level $X_1 = 1$, changing X_2 has **no influence** on the expected response, whereas for $j \geq 2$, changing X_2 **does** have an influence.

NOTE: A single m-way interaction term may correspond to **many** individual parameters. The general rules for counting parameters for main effects and interactions in a factorial model are established as follows:

Term	Type		No. of Levels	No. of Extra Parameters
Intercept				1
Main effect	Factor	X_1	M	$M-1$
	Covariate			1
2-way interaction	Two Factors	X_1, X_2	M_1, M_2	$(M_1-1)(M_2-1)$
	Two covariates			1
	Factor and covariate			$(M_1-1).1$
3-way interaction	Three Factors	X_1, X_2, X_3	M_1, M_2, M_2	$(M_1-1)(M_2-1)(M_3-1)$
	Three covariates			1

and so on.