MATH 523: EXPANSIONS FOR POISSON MODEL

Let $Y \sim Poisson(\mu)$, and let $g(Y)$ be a differentiable transformation. A Taylor expansion around μ yields

$$g(Y) = g(\mu) + (Y-\mu)\dot{g}(\mu) + \frac{1}{2}(Y-\mu)^2\ddot{g}(\mu) + \mathrm{O}_p((Y-\mu)^3)$$

Truncating at the first order term, as $\mathbb{E}[Y] = \mathrm{Var}[Y] = \mu$,

$$\mathbb{E}[g(Y)] \doteq g(\mu) \qquad \mathrm{Var}[g(Y)] = \{\dot{g}(\mu)\}^2\mu$$

Truncating at the second term

$$\mathbb{E}[g(Y)] \doteq g(\mu) + \frac{1}{2}\mathbb{E}[(Y-\mu)^2]\ddot{g}(\mu) = g(\mu) + \frac{\mu}{2}\ddot{g}(\mu)$$

and

$$\mathrm{Var}[g(Y)] \doteq \mathrm{Var}[g(\mu) + (Y-\mu)\dot{g}(\mu) + (Y-\mu)^2\ddot{g}(\mu)/2] = \mathrm{Var}[\dot{g}(\mu)Y + (Y-\mu)^2\ddot{g}(\mu)/2]$$

Now, using the formula for variance $\mathrm{Var}[X] = \mathbb{E}[X^2] - \{\mathbb{E}[X]\}^2$, for the first term

$$(\dot{g}(\mu)Y + (Y-\mu)^2\ddot{g}(\mu)/2)^2 = (\dot{g}(\mu)Y + (Y^2 - 2Y + \mu^2)\ddot{g}(\mu)/2)^2 = (a_0 + a_1Y + a_2Y^2)^2$$

where

$$a_0 = \mu^2\frac{\{\ddot{g}(\mu)\}^2}{4} \qquad a_1 = \dot{g}(\mu) - \frac{\{\ddot{g}(\mu)\}^2}{2} \qquad a_2 = \frac{\{\ddot{g}(\mu)\}^2}{4}$$

and

$$(a_0 + a_1Y + a_2Y^2)^2 = b_0 + b_1Y + b_2Y^2 + b_3Y^3 + b_4Y^4$$

where

$$b_0 = a_0^2 \quad b_1 = 2a_0a_1 \quad b_2 = 2a_0a_2 + a_1^2 \quad b_3 = 2a_1a_2 \qquad b_4 = a_2^2.$$

Thus to compute the $\mathbb{E}[X^2]$ term, we need $\mathbb{E}[Y^r]$ for $r = 1, 2, 3, 4$. We have

$$\begin{aligned}
\mathbb{E}[Y] &= \mu \\
\mathbb{E}[Y^2] &= \mu(1+\mu) \\
\mathbb{E}[Y^3] &= \mu(1+3\mu+\mu^2) \\
\mathbb{E}[Y^4] &= \mu(1+7\mu+6\mu^2+\mu^3)
\end{aligned}$$

For the second term, $\{\mathbb{E}[X]\}^2$, we have

$$\mathbb{E}[(\dot{g}(\mu)Y + (Y-\mu)^2\ddot{g}(\mu)/2] = \dot{g}(\mu)\mu + \frac{\mu\ddot{g}(\mu)}{2}$$

which we need to square.

Square-root transformation: $g(t) = t^{1/2}$

$$\dot{g}(t) = \frac{1}{2}t^{-1/2} \qquad \ddot{g}(t) = -\frac{1}{4}t^{-3/2}$$

so therefore

$$a_0 = \frac{1}{32\mu} \qquad a_1 = \frac{1}{2\mu^{1/2}} - \frac{1}{16\mu^3} \qquad a_2 = \frac{1}{32\mu^3}$$