

MATH 523 - MIDTERM 2013: SOLUTIONS

(a) We have for $y > 0$

$$\begin{aligned} f_Y(y; \mu, \sigma^2) &= \exp \left\{ -\frac{(y^2 - 2y\mu + \mu^2)}{2\mu^2 y \sigma^2} - \frac{1}{2} \log(2\pi \sigma^2 y^3) \right\} \\ &= \exp \left\{ \frac{y(-1/2\mu^2) + (1/\mu)}{\sigma^2} - \frac{1}{2y\sigma^2} - \frac{1}{2} \log(2\pi \sigma^2 y^3) \right\} \\ &= \exp \left\{ \frac{y\theta - b(\theta)}{\phi} + c(y, \phi) \right\} \end{aligned}$$

say, where

$$\theta = -\frac{1}{2\mu^2} \quad \phi = \sigma^2$$

is the canonical parameter, and $b(\mu) = -1/\mu$. Note that

$$\mu = (-2\theta)^{-1/2} \quad \therefore \quad b(\theta) = -(-2\theta)^{1/2}$$

and hence

$$\dot{b}(\theta) = -(1/2) \times (-2\theta)^{-1/2} \times (-2) = (-2\theta)^{-1/2} = \mu$$

as required.

6 MARKS

(b) We have

$$\ddot{b}(\theta) = (-1/2) \times (-2\theta)^{-3/2} \times (-2) = (-2\theta)^{-3/2} = \mu^3$$

and hence the variance is $\sigma^2 \mu^3$, with $V(\mu) = \mu^3$

6 MARKS

(c) Canonical link is $g(t) = -t^{-2}$; as μ is positive, we might also use the log link $g(t) = \log t$ or the identity link $g(t) = t$. Here the canonical link does not seem very sensible; it is monotonically increasing, but as the canonical parameter is **negative**, would need some constraints on the β s. Of the ones suggested, the log link is most appropriate.

6 MARKS

(d) The log transform is the most sensible choice, as this always removes positive skewness at least in part. Furthermore, using the logic of the Anscombe transformation, we have for $A(t)$

$$A(t) = \int_0^t V^{-1/3}(u) du = \int_0^t u^{-1} du = \log t$$

and we know that the Anscombe transform removes skewness.

4 MARKS

(e) We have

$$l_n(\beta) = \frac{1}{\sigma^2} \sum_{i=1}^n \left[yx_i\beta + (-2x_i\beta)^{1/2} \right] + \text{const}$$

so therefore

$$\frac{\partial l_n(\beta)}{\partial \beta} = \frac{1}{\sigma^2} \sum_{i=1}^n x_i^\top \left[y_i - (-2x_i\beta)^{-1/2} \right] = \frac{1}{\sigma^2} \sum_{i=1}^n x_i^\top (y_i - \mu_i) \quad (p \times 1)$$

yields the p equations.

6 MARKS

- (f) Under the full model, $\hat{\mu}_i = y_i$. Therefore the deviance is

$$D(\mathbf{y}; \hat{\boldsymbol{\mu}}) = 2 \sum_{i=1}^n \left[y_i \left\{ -\frac{1}{2y_i^2} + \frac{1}{2\hat{\mu}_i^2} \right\} - \left\{ -\frac{1}{y_i} + \frac{1}{\hat{\mu}_i} \right\} \right]$$

6 MARKS

- (g) There are two models fitted: the first (lines 13-31) includes the binary factor, but from the coefficients table, there is no impact of the factor (line 24 shows no significance). The second model (lines 33-52) fits the null model. The differences in deviance are negligible even after rescaling by the dispersion indicating, in summary, that there is no difference between AM and PM shifts. Finally, using the estimated dispersion, we have for the null model that

$$\frac{D(\mathbf{y}; \hat{\boldsymbol{\mu}})}{\hat{\phi}} = \frac{4.2861}{0.118} \simeq 36.32$$

on 30 residual degrees of freedom, so apparently a reasonable fit, that is, the inverse Gaussian model is adequate.

8 MARKS

- (h) Inspecting the null model, we see (line 43) that the estimated intercept is 0.002768. The pooled data mean, however, is (line 10) 19.00645. Therefore, by inspection, we see that

$$1/(19.00645^2) = 0.002768$$

so the default link must be $g(t) = 1/t^2$.

5 MARKS

- (i) This is a summary of the deviance residuals, as defined in lectures, for this data set.

3 MARKS