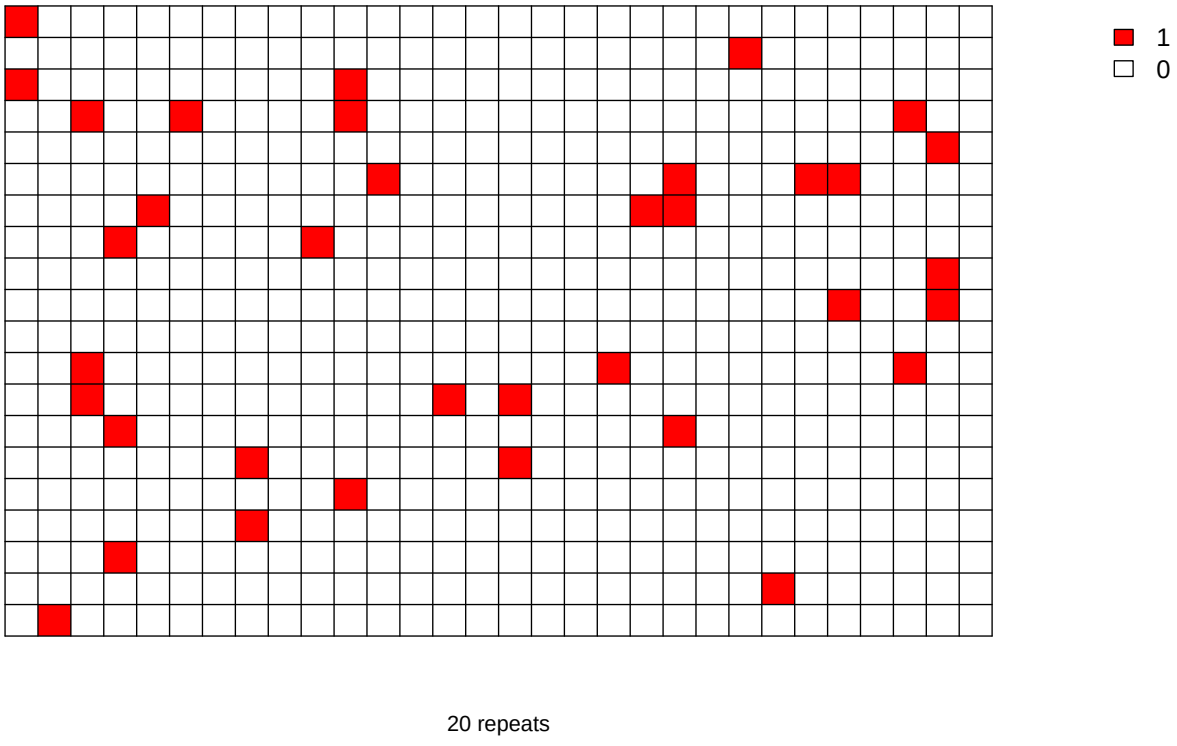
THE POISSON DISTRIBUTION

The Poisson distribution arises as the limiting case of the Binomial distribution whose parameter p varies with n. Specifically, we consider a fixed constant λ, and consider the $Binomial(n, p)$ distribution, and observe what happens when n increases with $p = \lambda/n$ decreasing so that $np = \lambda$ stays fixed. We have that as $n \longrightarrow \infty$

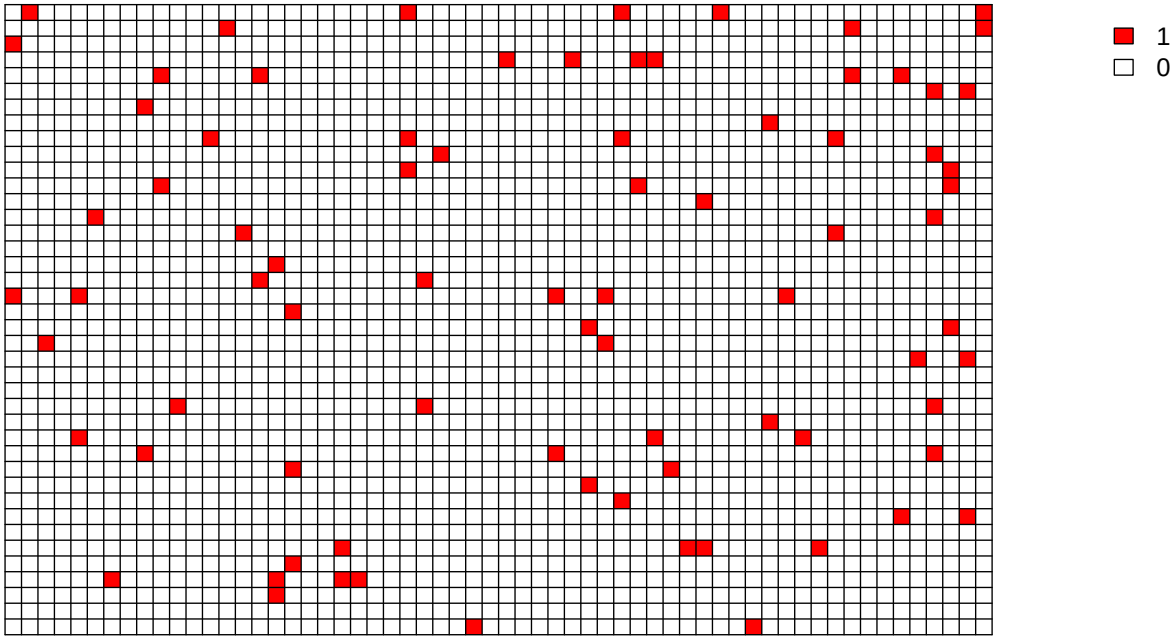
$$\binom{n}{y}\left(\frac{\lambda}{n}\right)^y\left(1-\frac{\lambda}{n}\right)^{n-y} \longrightarrow e^{-\lambda}\frac{\lambda^y}{y!}$$

In the following plots, we generate binary sequences repeated a number of times (repeats are in the rows of the plots), with 1s or 'successes' marked as the red squares, and 0s or 'failures' marked as white squares. In the simulations, $\lambda = 2$ is fixed, and $n = 30, 60, \ldots, 300$.

```
lam<-2
 par(mar=c(5.1, 4.1, 4.1, 8.1), xpd=TRUE)
for(n in 1:10){
        xn<-30*n
        yn<-20*n
        p<-lam/xn
        xmat<-matrix(rbinom(xn*yn,1,p),nrow=xn,ncol=yn)
        image((1:xn/xn)-0.5/xn,(1:yn)/yn-0.5/yn,xmat,xlab=substitute(paste(nval,' repeats'),
            list(nval = yn)),ylab='',axes=FALSE,col=rev(rgb(c(1,1),c(0,1),c(0,1))))
        for(i in 0:yn){lines(c(0,1),c(i,i)/yn)}
        for(i in 0:xn){lines(c(i,i)/xn,c(0,1))}
        title(substitute(paste(n==nval,' ',p==pval), list(nval = xn, pval=p)))
        legend(1.1,1.0,c('1','0'),fill=rgb(c(1,1),c(0,1),c(0,1)),cex=1.2,bty='n')
}
```

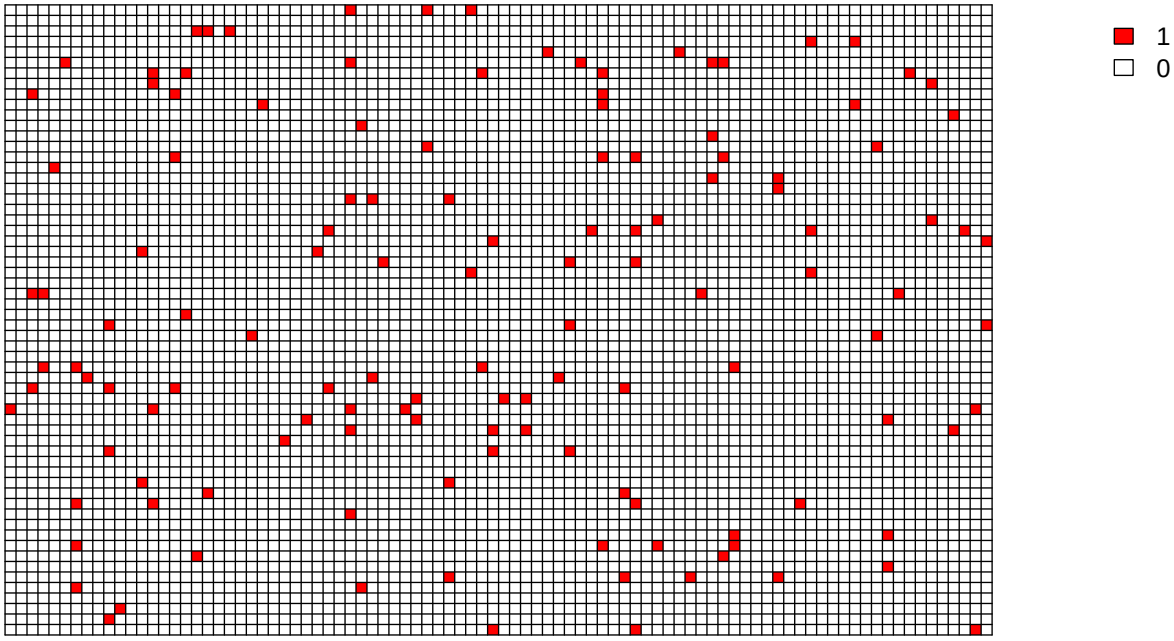


n = 60 p = 0.03333333



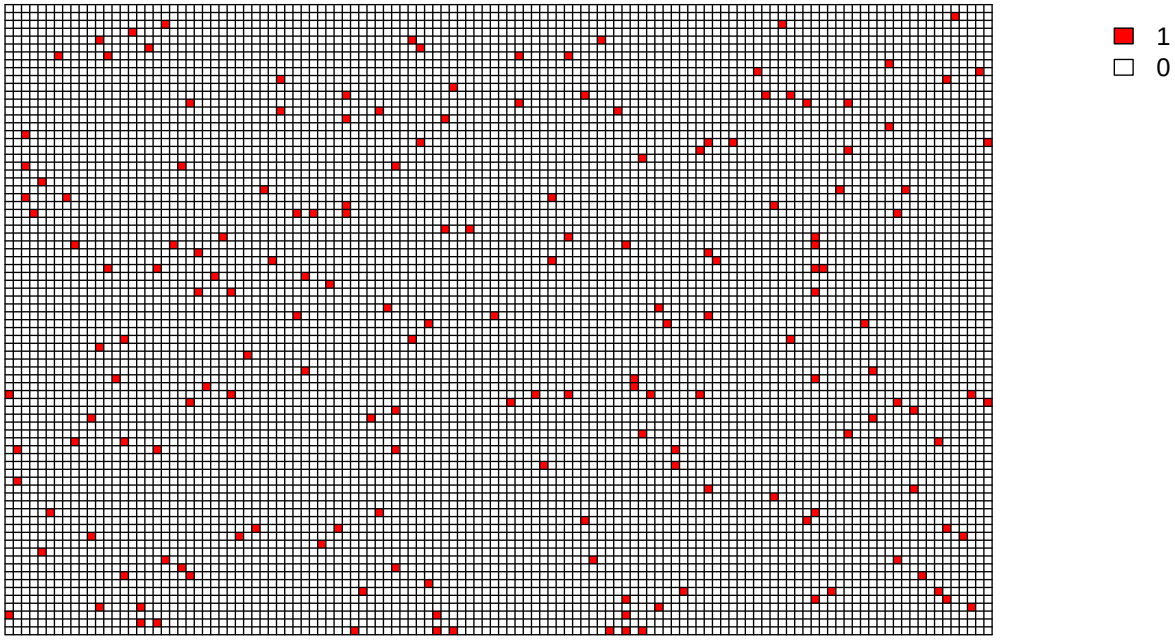
40 repeats

n = 90 p = 0.02222222



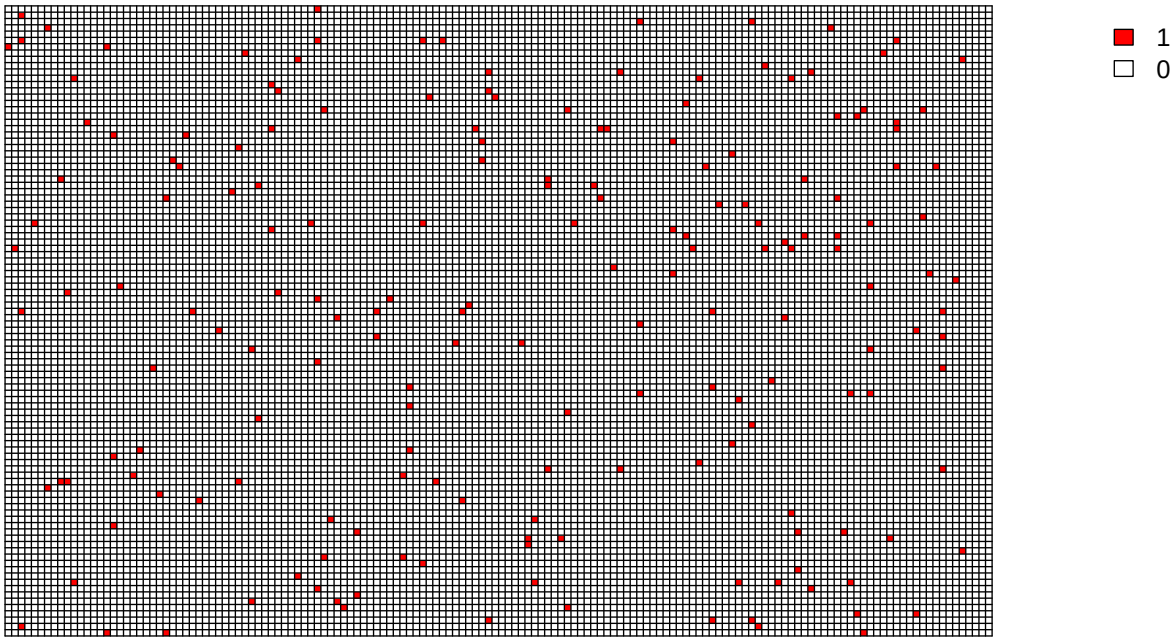
60 repeats

n = 120 p = 0.01666667



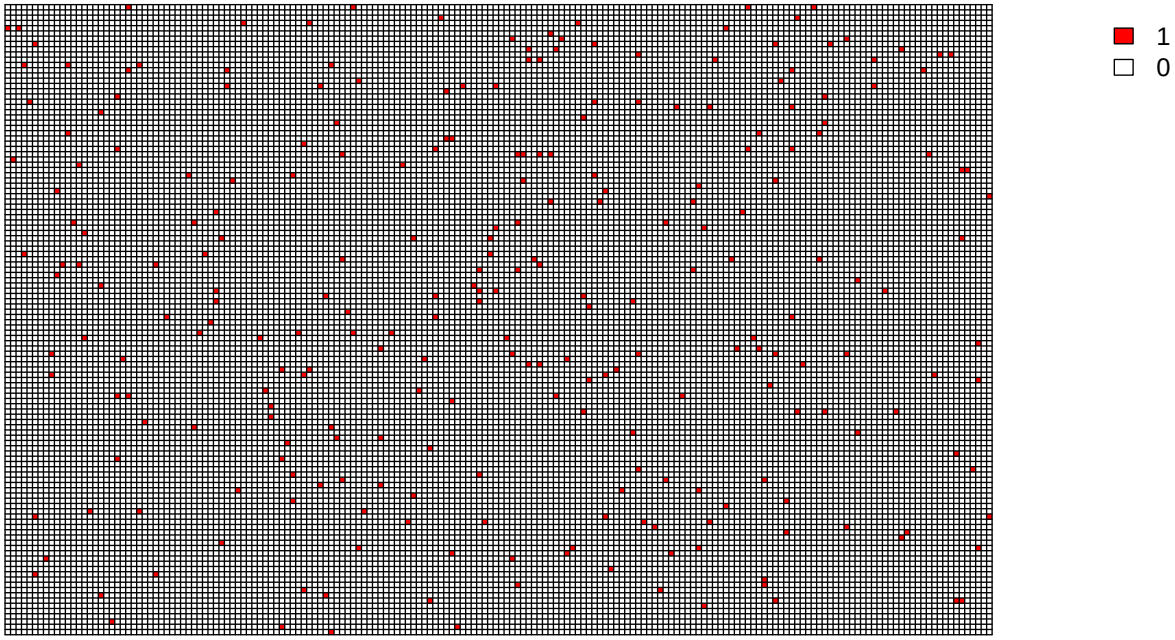
80 repeats

n = 150 p = 0.01333333



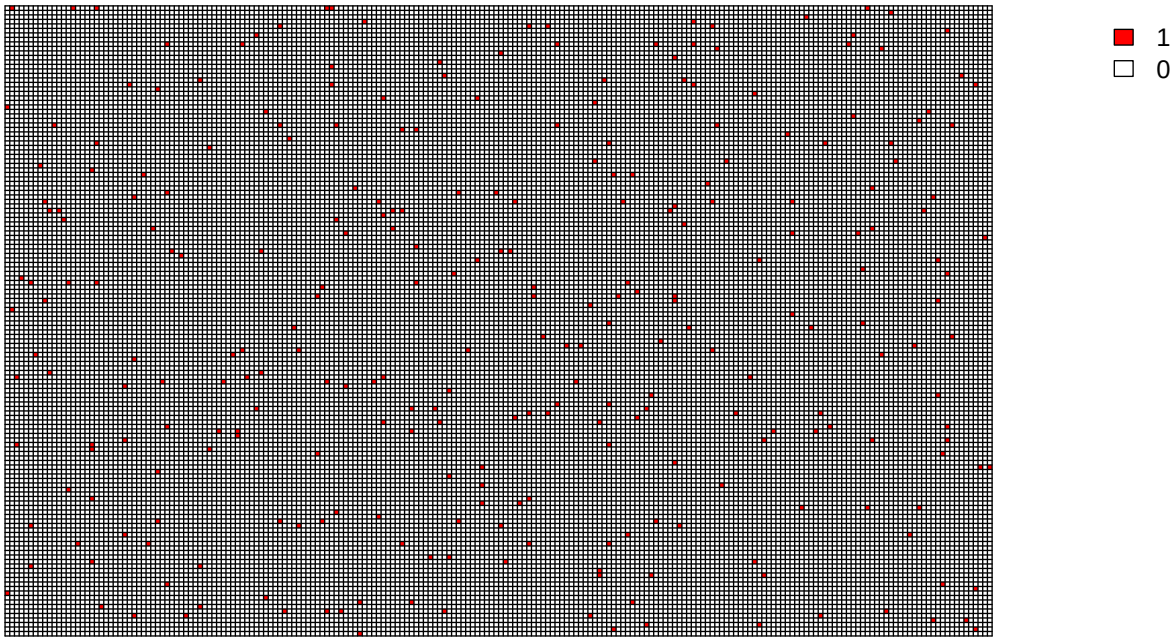
100 repeats

n = 180 p = 0.01111111



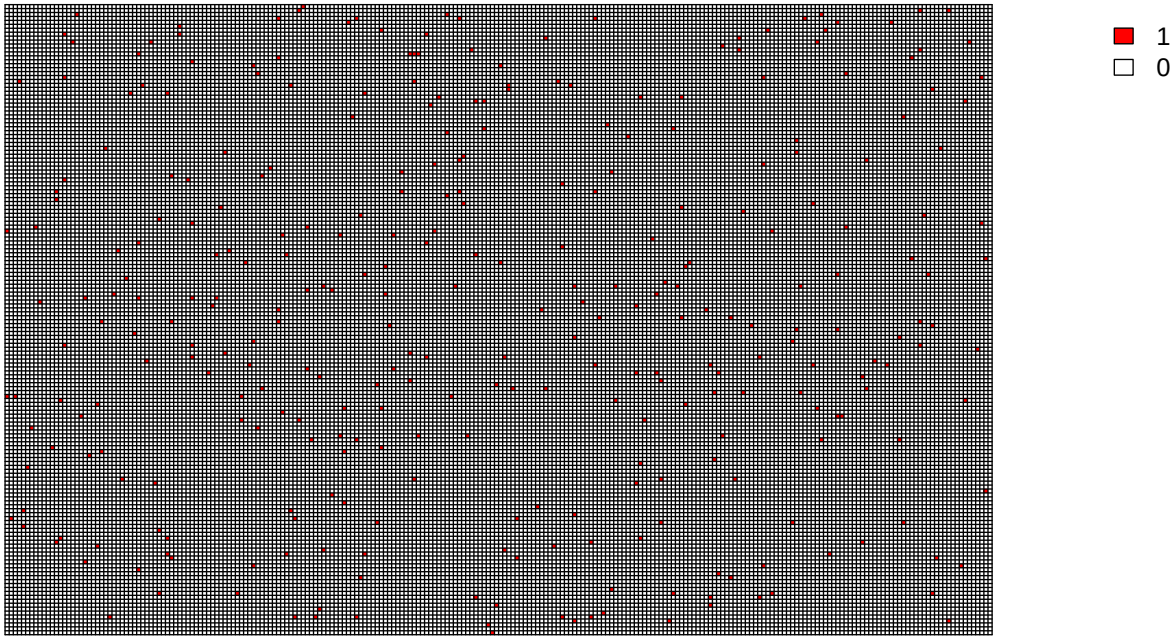
120 repeats

n = 210 p = 0.00952381



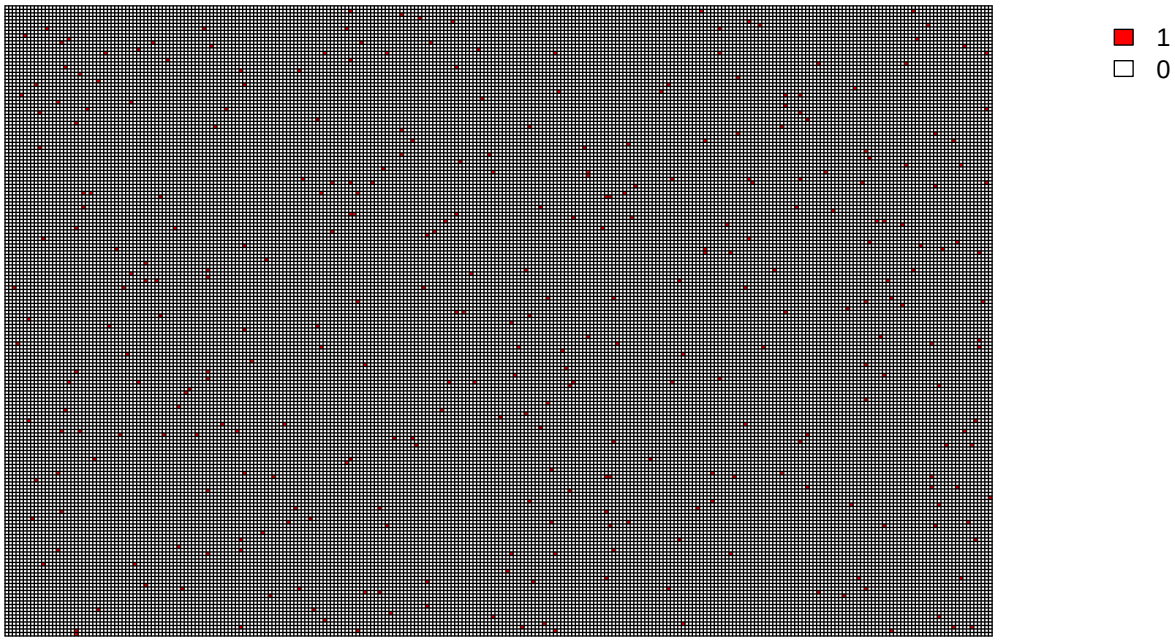
140 repeats

n = 240 p = 0.008333333

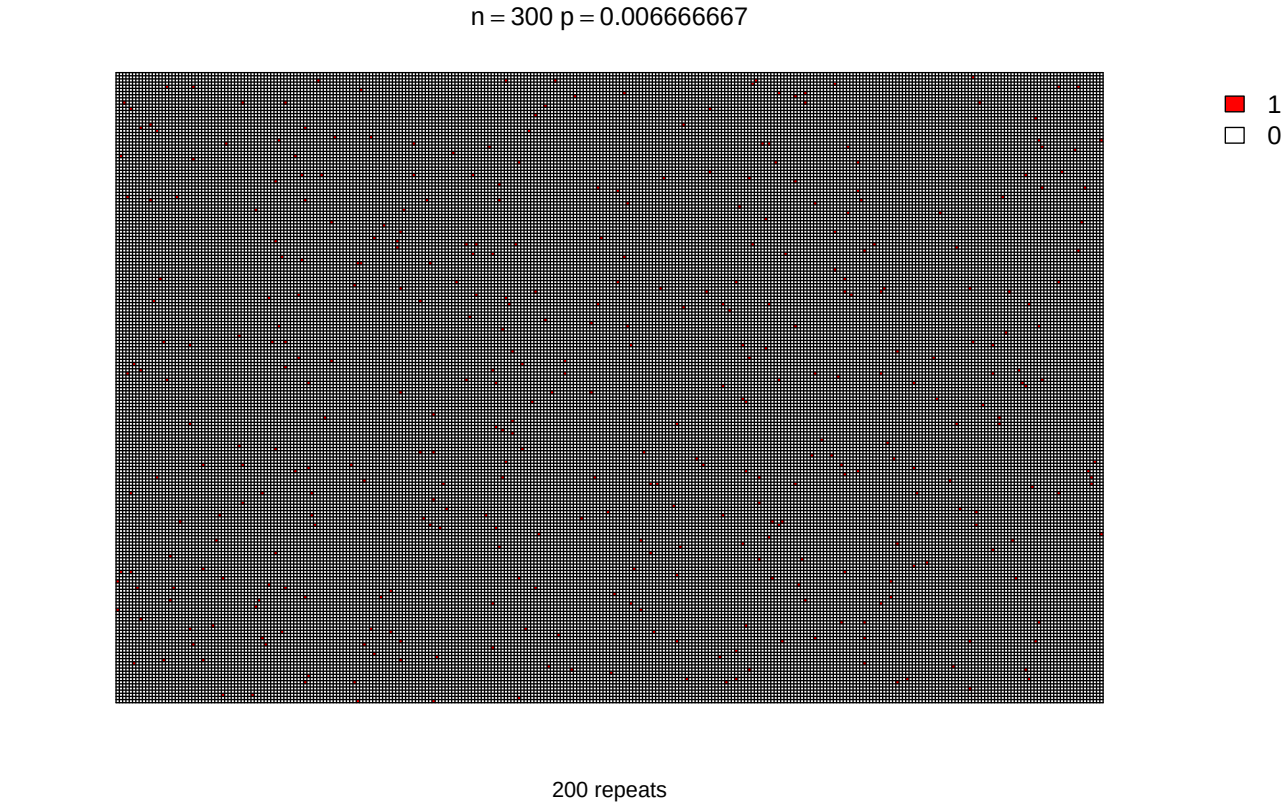


160 repeats

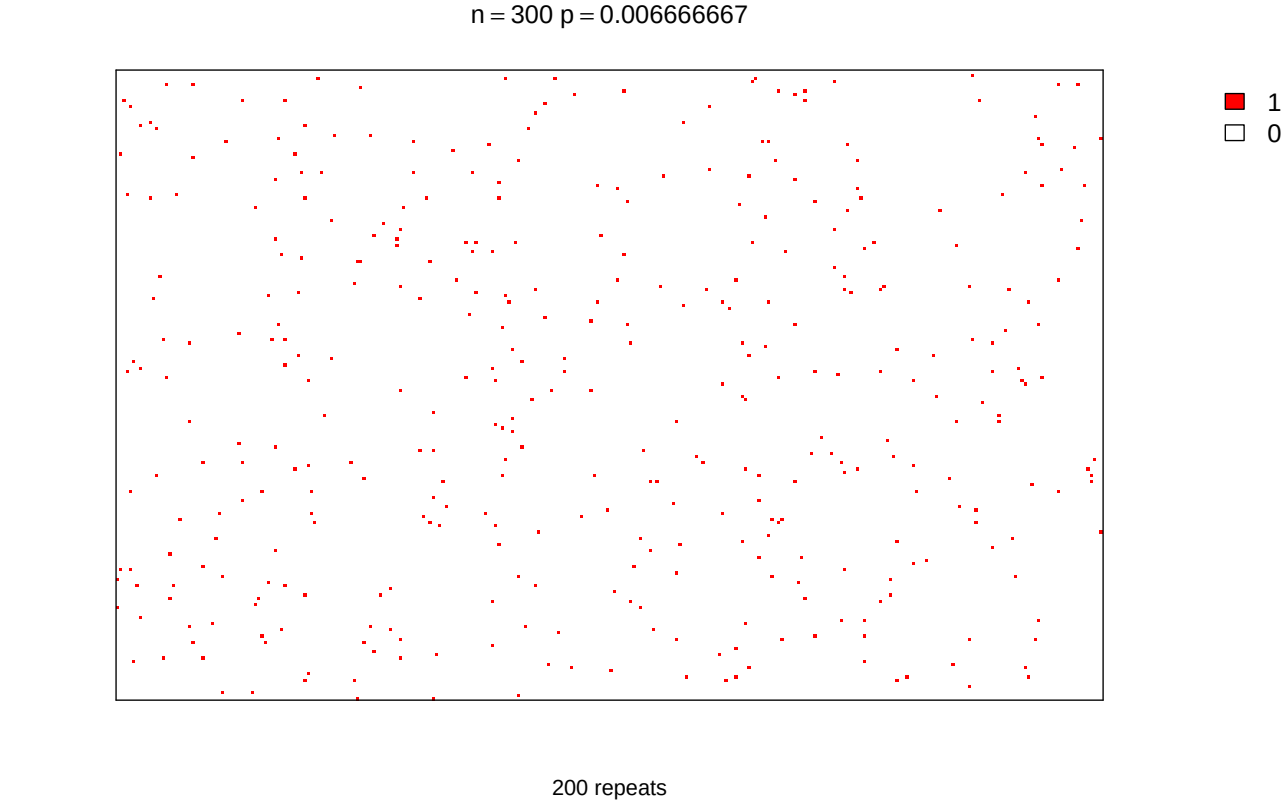
n = 270 p = 0.007407407



180 repeats



We remove the gridlines to make the plot easier to see.



We can investigate how well the Poisson distribution approximates the binomial distribution: in each case, $p = \lambda/n$, with $\lambda = 2$. Entries in the table are

$$p(y) = P(Y = y) \qquad y = 0, 1, 2, \ldots, 9$$

for n up to 600.

```
+            y=0    y=1    y=2    y=3    y=4    y=5    y=6    y=7    y=8    y=9
+ n=30    0.1262 0.2705 0.2801 0.1867 0.0900 0.0334 0.0100 0.0024 0.0005 0.0001
+ n=60    0.1308 0.2706 0.2753 0.1835 0.0902 0.0348 0.0110 0.0029 0.0007 0.0001
+ n=90    0.1323 0.2706 0.2737 0.1825 0.0902 0.0353 0.0114 0.0031 0.0007 0.0002
+ n=120   0.1331 0.2707 0.2730 0.1820 0.0902 0.0355 0.0115 0.0032 0.0008 0.0002
+ n=150   0.1335 0.2707 0.2725 0.1817 0.0902 0.0356 0.0116 0.0032 0.0008 0.0002
+ n=180   0.1338 0.2707 0.2722 0.1815 0.0902 0.0357 0.0117 0.0033 0.0008 0.0002
+ n=210   0.1340 0.2707 0.2720 0.1813 0.0902 0.0357 0.0117 0.0033 0.0008 0.0002
+ n=240   0.1342 0.2707 0.2718 0.1812 0.0902 0.0358 0.0118 0.0033 0.0008 0.0002
+ n=270   0.1343 0.2707 0.2717 0.1811 0.0902 0.0358 0.0118 0.0033 0.0008 0.0002
+ n=300   0.1344 0.2707 0.2716 0.1811 0.0902 0.0358 0.0118 0.0033 0.0008 0.0002
+ n=330   0.1345 0.2707 0.2715 0.1810 0.0902 0.0359 0.0118 0.0033 0.0008 0.0002
+ n=360   0.1346 0.2707 0.2714 0.1810 0.0902 0.0359 0.0119 0.0034 0.0008 0.0002
+ n=390   0.1346 0.2707 0.2714 0.1809 0.0902 0.0359 0.0119 0.0034 0.0008 0.0002
+ n=420   0.1347 0.2707 0.2713 0.1809 0.0902 0.0359 0.0119 0.0034 0.0008 0.0002
+ n=450   0.1347 0.2707 0.2713 0.1808 0.0902 0.0359 0.0119 0.0034 0.0008 0.0002
+ n=480   0.1348 0.2707 0.2712 0.1808 0.0902 0.0359 0.0119 0.0034 0.0008 0.0002
+ n=510   0.1348 0.2707 0.2712 0.1808 0.0902 0.0359 0.0119 0.0034 0.0008 0.0002
+ n=540   0.1348 0.2707 0.2712 0.1808 0.0902 0.0360 0.0119 0.0034 0.0008 0.0002
+ n=570   0.1349 0.2707 0.2711 0.1808 0.0902 0.0360 0.0119 0.0034 0.0008 0.0002
+ n=600   0.1349 0.2707 0.2711 0.1807 0.0902 0.0360 0.0119 0.0034 0.0008 0.0002
+ Poisson 0.1353 0.2707 0.2707 0.1804 0.0902 0.0361 0.0120 0.0034 0.0009 0.0002
```

The probabilities match to the third decimal place.