JOINT CONTINUOUS DISTRIBUTIONS

Joint cdf: If Y_1 and Y_2 are continuous random variables, then we may consider the probability

$$F_{Y_1,Y_2}(y_1, y_2) = P((Y_1 \leq y_1) \cap (Y_2 \leq y_2)) = P(Y_1 \leq y_1, Y_2 \leq y_2)$$

for any pair of real numbers (y_1, y_2). The joint cdf has the following properties:

- "starts at zero"

$$\lim_{y_1 \longrightarrow -\infty} \lim_{y_2 \longrightarrow -\infty} F_{Y_1,Y_2}(y_1, y_2) = 0$$

$$\lim_{y_1 \longrightarrow -\infty} F_{Y_1,Y_2}(y_1, y_2) = 0 \quad \text{for all } y_2 \qquad \text{and} \qquad \lim_{y_2 \longrightarrow -\infty} F_{Y_1,Y_2}(y_1, y_2) = 0 \quad \text{for all } y_1$$

- "ends at one"

$$\lim_{y_1 \longrightarrow \infty} \lim_{y_2 \longrightarrow \infty} F_{Y_1,Y_2}(y_1, y_2) = 1$$

- "non-decreasing in y_1 and y_2 in between"

$$F_{Y_1,Y_2}(y_1, y_2) \leq F_{Y_1,Y_2}(y_1 + c, y_2)$$

$$F_{Y_1,Y_2}(y_1, y_2) \leq F_{Y_1,Y_2}(y_1, y_2 + c)$$

  for all y_1, y_2, and any $c > 0$.

Furthermore, we have that

$$\lim_{y_1 \longrightarrow \infty} F_{Y_1,Y_2}(y_1, y_2) = F_{Y_2}(y_2) \qquad \lim_{y_2 \longrightarrow \infty} F_{Y_1,Y_2}(y_1, y_2) = F_{Y_1}(y_1).$$

EXAMPLE: Consider the function defined

- for $0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1$ by

$$F_{Y_1,Y_2}(y_1, y_2) = \frac{4}{5}\left(y_1 y_2 + \frac{y_1^2 y_2^2}{4}\right)$$

- for $y_1 < 0$ or $y_2 < 0$ by

$$F_{Y_1,Y_2}(y_1, y_2) = 0$$

- for $y_1 > 1$, but $0 \leq y_2 \leq 1$ by

$$F_{Y_1,Y_2}(y_1, y_2) = F_{Y_1,Y_2}(1, y_2) = \frac{4}{5}\left(y_2 + \frac{y_2^2}{4}\right)$$

  which follows as

$$P(Y_1 \leq y_1, Y_2 \leq y_2) = P(Y_1 \leq 1, Y_2 \leq y_2).$$

- for $y_2 > 1$, but $0 \leq y_1 \leq 1$ by

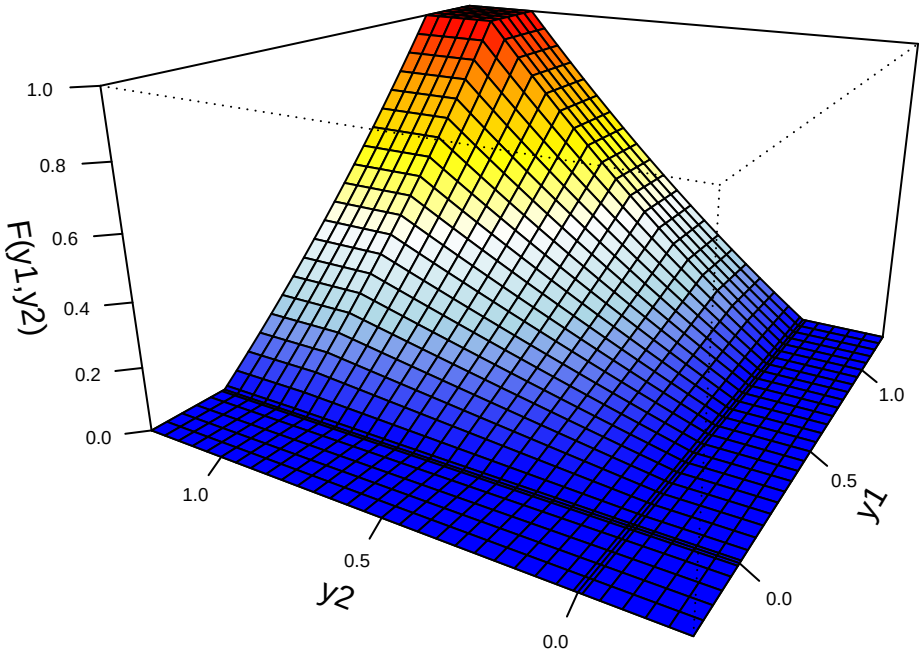
$$F_{Y_1,Y_2}(y_1, y_2) = F_{Y_1,Y_2}(y_1, 1) = \frac{4}{5}\left(y_1 + \frac{y_1^2}{4}\right)$$

  which follows as in this region

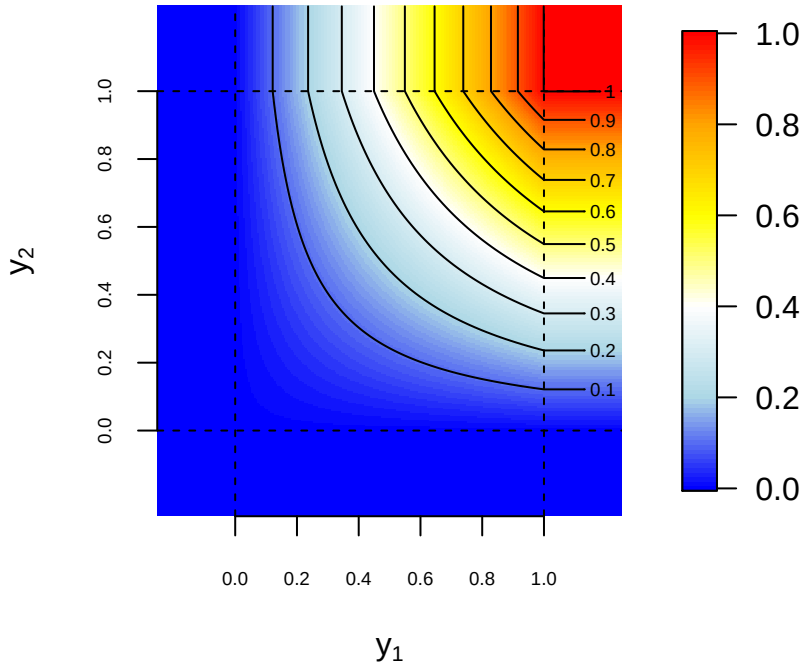
$$P(Y_1 \leq y_1, Y_2 \leq y_2) = P(Y_1 \leq y_1, Y_2 \leq 1).$$

- for $y_1 > 1$ and $y_2 > 1$ by $F_{Y_1,Y_2}(y_1, y_2) = 1$.

This function defines a joint cdf for two continuous random variables Y_1 and Y_2 taking values on $[0,1] \times [0,1]$.



This surface plot displays the function $F_{Y_1,Y_2}(y_1, y_2)$ on a grid of values. The function is zero in the dark blue region, and increases in both y_1 and y_2 for values in the range $(0, 1)$. It is also possible to plot the function as an image plot (see below), with contours linking points on the plot where the function is constant, that is, where $F_{Y_1,Y_2}(y_1, y_2) = p$, for values p, $0 \leq p \leq 1$.



Joint pdf: We then use a similar construction to the single variable case to define the joint pdf, that is, we write

$$F_{Y_1,Y_2}(y_1,y_2) = \int_{-\infty}^{y_1}\int_{-\infty}^{y_2} f_{Y_1,Y_2}(t_1,t_2)\,dt_2dt_1$$

to define $f_{Y_1,Y_2}(y_1,y_2)$ implicitly. We have then that

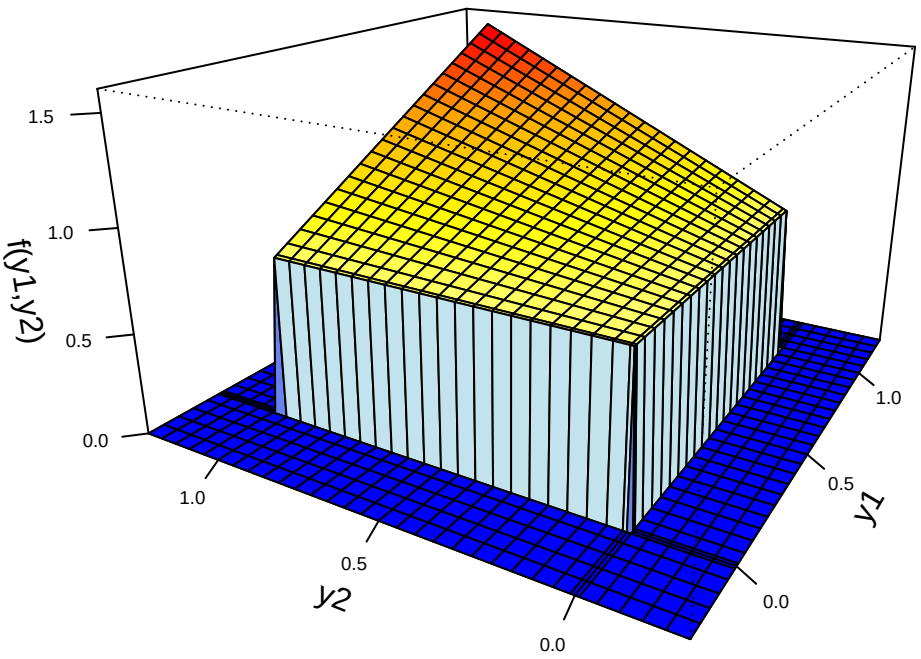
$$f_{Y_1,Y_2}(y_1,y_2) = \frac{\partial^2}{\partial y_1 \partial y_2}\left\{F_{Y_1,Y_2}(y_1,y_2)\right\}$$

In the above example, we have that for $0 \le y_1 \le 1$, $0 \le y_2 \le 1$

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) = \frac{\partial^2}{\partial y_1 \partial y_2}\left\{\frac{4}{5}\left(y_1y_2 + \frac{y_1^2y_2^2}{4}\right)\right\} &= \frac{\partial}{\partial y_1}\left\{\frac{\partial}{\partial y_2}\left\{\frac{4}{5}\left(y_1y_2 + \frac{y_1^2y_2^2}{4}\right)\right\}\right\} \\ &= \frac{\partial}{\partial y_1}\left\{\frac{4}{5}\left(y_1 + \frac{y_1^2y_2}{2}\right)\right\} \\ &= \frac{4}{5}\left(1 + y_1y_2\right) \end{aligned}$$

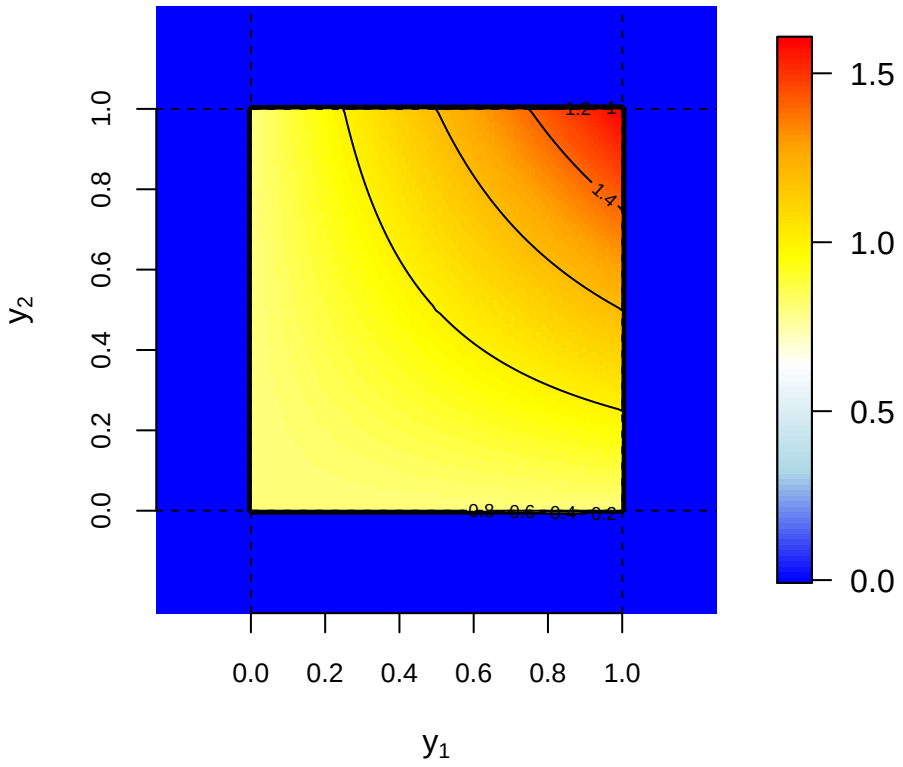
Notice that for y_1 and y_2 outside of the range $0 \le y_1 \le 1$, $0 \le y_2 \le 1$, we have that

$$f_{Y_1,Y_2}(y_1,y_2) = 0.$$



This surface plot of the joint pdf indicates that the joint pdf in this case is also increasing in y_1 and y_2 on the

range $(0, 1)$ for each variable.



Marginal distributions: we may compute the marginal pdfs by noting that

$$F_{Y_1}(y_1) = F_{Y_1,Y_2}(y_1, 1) = \frac{4}{5}\left(y_1 + \frac{y_1^2}{4}\right) \qquad 0 \le y_1 \le 1$$

with $F_{Y_1}(y_1) = 0$ if $y_1 < 0$ and $F_{Y_1}(y_1) = 1$ if $y_1 > 0$. Similarly

$$F_{Y_2}(y_2) = F_{Y_1,Y_2}(1, y_2) = \frac{4}{5}\left(y_2 + \frac{y_2^2}{4}\right) \qquad 0 \le y_2 \le 1$$

with $F_{Y_2}(y_2) = 0$ if $y_2 < 0$ and $F_{Y_2}(y_2) = 1$ if $y_2 > 0$. By differentiation, we have that

$$f_{Y_1}(y_1) = \begin{cases} \frac{4}{5}\left(1 + \frac{y_1}{2}\right) & 0 \le y_1 \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

Notice that

$$\int_{-\infty}^{\infty} f_{Y_1}(y_1)\, dy_1 = \int_0^1 \frac{4}{5}\left(1 + \frac{y_1}{2}\right)\, dy_1 = \frac{4}{5}\left[y_1 + \frac{y_1^2}{4}\right]_0^1 = \frac{4}{5}\left[1 + \frac{1}{4}\right] = 1.$$

By an identical calculation

$$f_{Y_2}(y_2) = \begin{cases} \frac{4}{5}\left(1 + \frac{y_2}{2}\right) & 0 \le y_2 \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

Here we have that for general (y_1, y_2), $f_{Y_1,Y_2}(y_1, y_2) \neq f_{Y_1}(y_1) f_{Y_2}(y_2)$, so therefore the random variables Y_1 and Y_2 are **not independent** (that is, they are **dependent**).

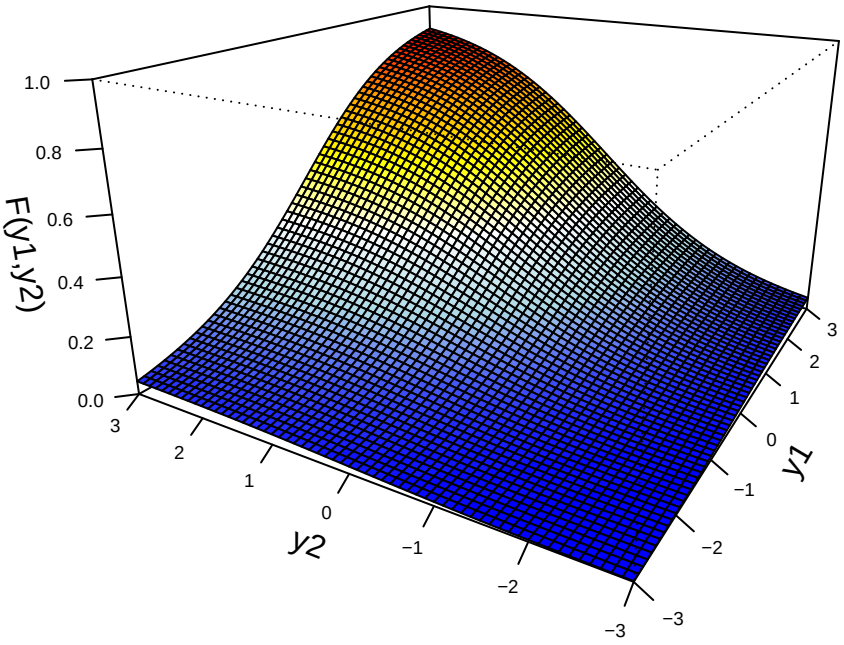
Conditional pdfs: we can compute the conditional pdfs using the definition. For example, for $0 \le y_2 \le 1$

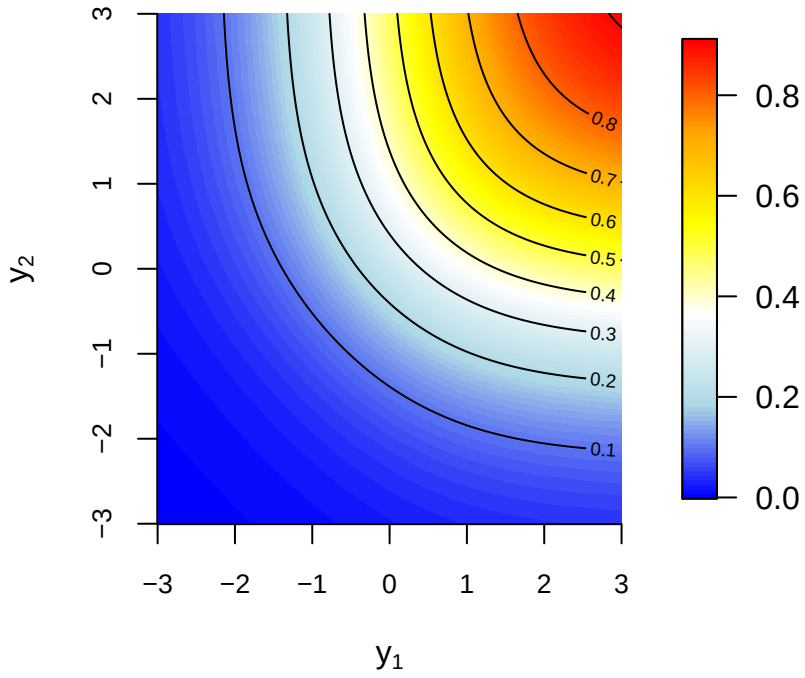
$$f_{Y_1|Y_2}(y_1|y_2) = \frac{f_{Y_1,Y_2}(y_1, y_2)}{f_{Y_2}(y_2)} = \frac{(1 + y_1 y_2)}{\left(1 + \frac{y_2}{2}\right)} \qquad 0 \le y_1 \le 1, \text{zero otherwise.}$$

EXAMPLE: Consider the function defined for $y_1 \in \mathbb{R}$ and $y_2 \in \mathbb{R}$ by

$$F_{Y_1,Y_2}(y_1, y_2) = \frac{e^{y_1+y_2}}{(1+e^{y_1})(1+e^{y_2})}$$

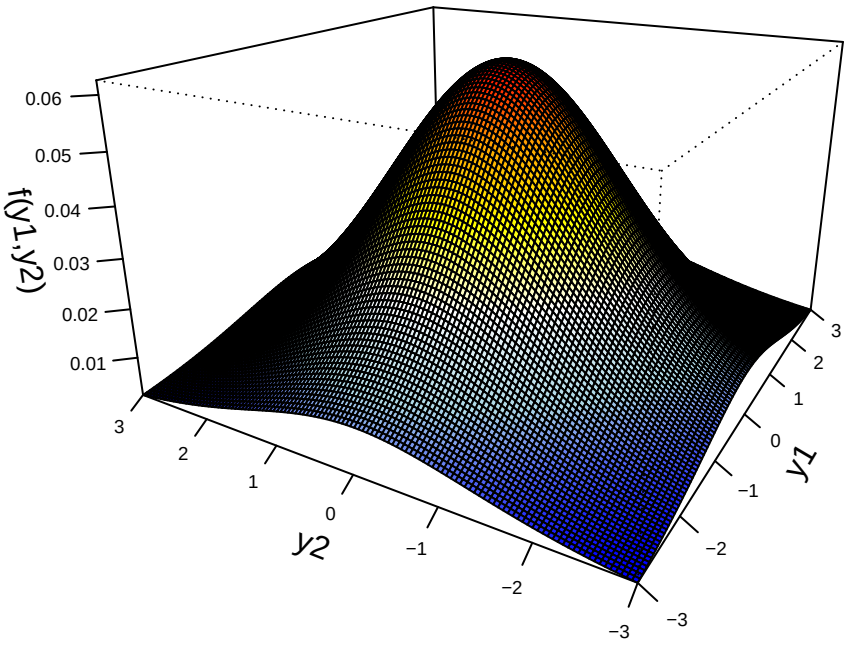
This function defines a joint cdf for two continuous random variables Y_1 and Y_2.

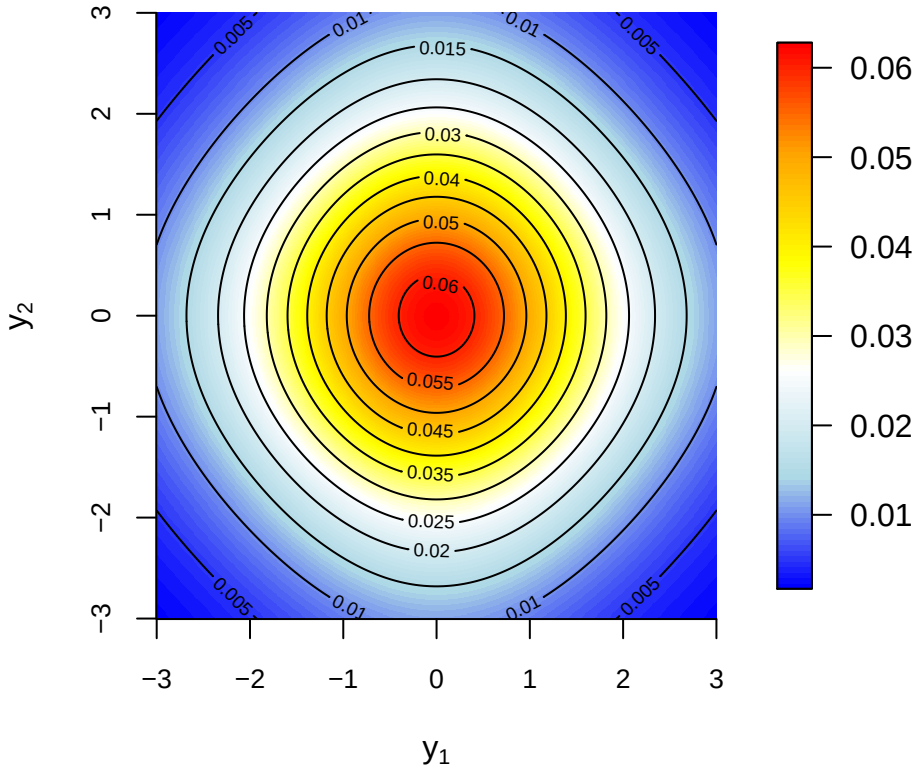




Joint pdf: We have by direct calculation, differentiating the joint cdf, that

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{e^{y_1+y_2}}{(1+e^{y_1})^2(1+e^{y_2})^2} = \frac{e^{y_1}}{(1+e^{y_1})^2}\frac{e^{y_2}}{(1+e^{y_2})^2}$$





Marginal distributions: we may compute the marginal pdfs by noting that

$$F_{Y_1}(y_1) = F_{Y_1,Y_2}(y_1, \infty) = \frac{e^{y_1}}{(1+e^{y_1})} \qquad y_1 \in \mathbb{R}$$

and

$$F_{Y_2}(y_2) = F_{Y_1,Y_2}(\infty, y_2) = \frac{e^{y_2}}{(1+e^{y_2})} \qquad y_2 \in \mathbb{R}$$

By differentiation, we have that

$$f_{Y_1}(y_1) = \frac{e^{y_1}}{(1+e^{y_1})^2} \qquad y_1 \in \mathbb{R}$$

and

$$f_{Y_2}(y_2) = \frac{e^{y_2}}{(1+e^{y_2})^2} \qquad y_2 \in \mathbb{R}$$

Here we have that for all (y_1, y_2)

$$f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2)$$

so therefore the random variables Y_1 and Y_2 are **independent**.

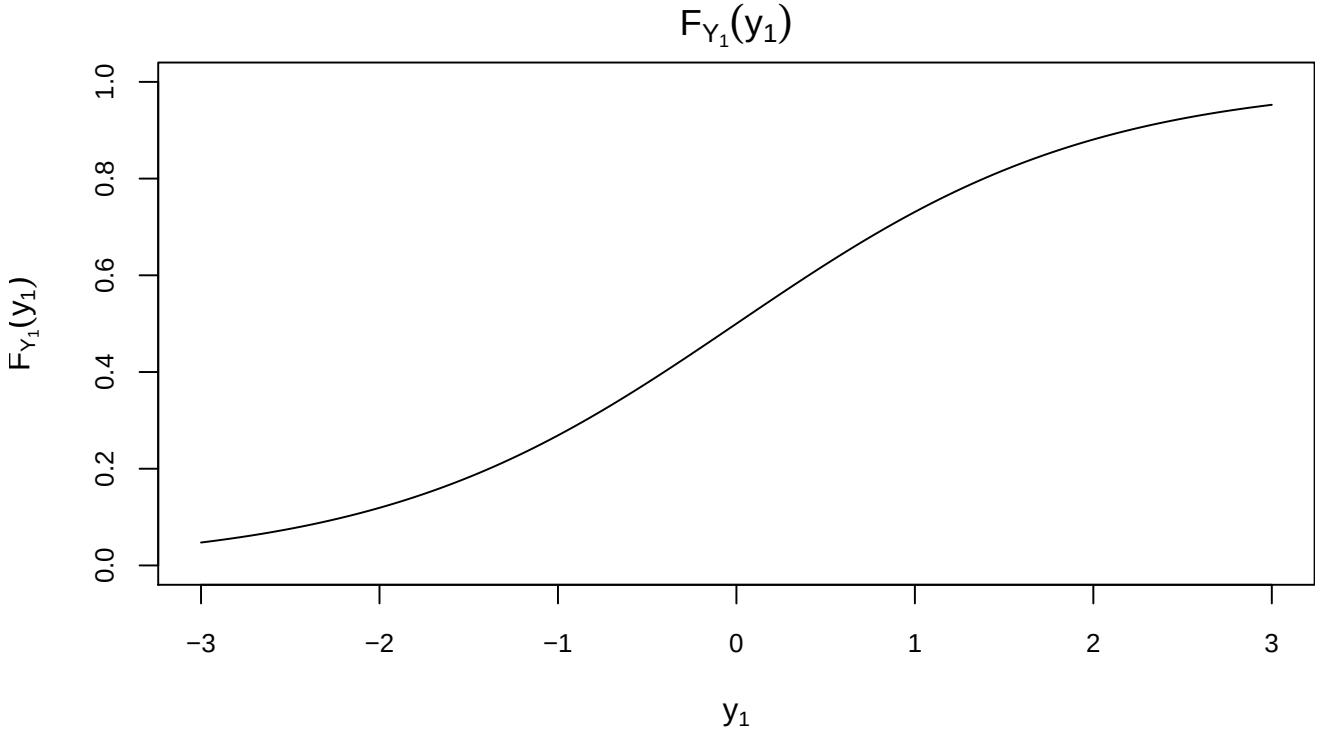
Conditional pdfs: Due to independence, we have that

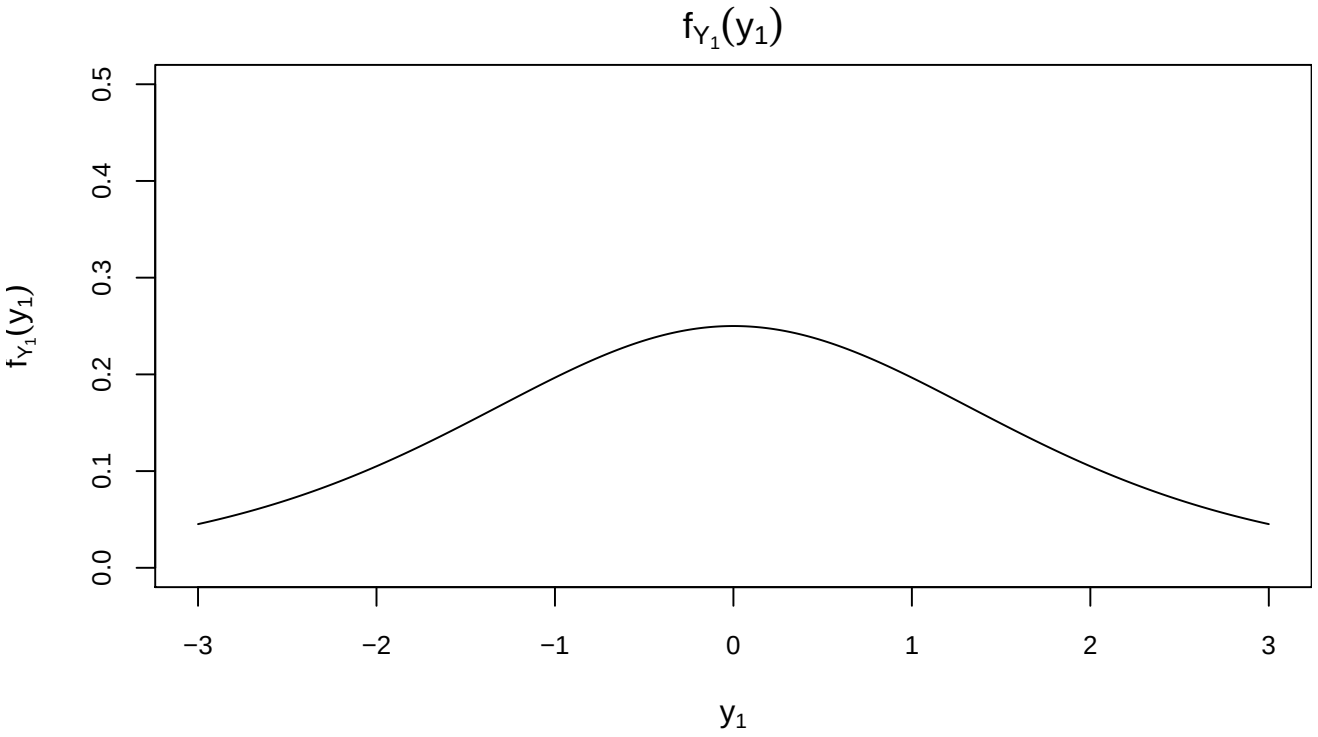
$$f_{Y_1|Y_2}(y_1|y_2) = f_{Y_1}(y_1)$$

for all y_1 and

$$f_{Y_2|Y_1}(y_2|y_1) = f_{Y_2}(y_2)$$

for all y_2.




$f_{Y_1}(y_1)$
$f_{Y_1}(y_1)$
0.0
0.1
0.2
0.3
0.4
0.5
−3
−2
−1
0
1
2
3
y_1