# Appendix

# Connection to the Geometric distribution

Recall the origin of the Poisson distribution; we consider

$$Y \sim Binomial(n, p)$$

then fix $\lambda = np$, and allow $n \longrightarrow \infty$.

- We have an infinite sequence of independent Bernoulli (binary) trials, where the successes occur at a constant rate of $\lambda$, independently of the results of other trials.
- Furthermore, the number of trials between successes are independent $Geometric(p)$ random variables

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# Connection to the Geometric distribution (cont.)

In the sequence of infinite independent Bernoulli trials, let $Y_1$ be the number of trials until the first success. Then

$$Y_1 \sim \mathit{Geometric}(p),$$

with

$$p_{Y_1}(y) = (1-p)^{y-1}p \qquad y = 1, 2, 3, \ldots$$

with $p_{Y_1}(y)$ otherwise, with corresponding cdf

$$F_{Y_1}(y) = \begin{cases} 0 & y < 1 \\ 1 - (1-p)^{\lfloor y \rfloor} & y \geq 1 \end{cases}$$

where $\lfloor y \rfloor$ is the largest integer no greater than $y$.

# Connection to the Geometric distribution (cont.)

Now, let $X_1 = Y_1/n$, and $\lambda = np$. Then for fixed $x$

$$F_{X_1}(x) = P(X_1 \leq x) = P(Y_1 \leq nx) = F_{Y_1}(nx)$$

that is

$$F_{X_1}(x) = \begin{cases} 0 & x < 1/n \\ 1 - \left(1 - \frac{\lambda}{n}\right)^{\lfloor nx \rfloor} & x \geq 1/n \end{cases}$$

# Connection to the Geometric distribution (cont.)

As $n \longrightarrow \infty$

$$\left(1 - \frac{\lambda}{n}\right)^{\lfloor nx \rfloor} \longrightarrow \exp\{-\lambda x\}$$

for $x > 0$. That is

$$F_{X_1}(x) \longrightarrow \begin{cases} 0 & x < 0 \\ 1 - \exp\{-\lambda x\} & x \geq 0 \end{cases}$$

Thus $X_1 \sim \mathit{Exponential}(1/\lambda)$.

# Connection to the Geometric distribution (cont.)

This result also follows using mgfs: we have that

$$m_{Y_1}(t) = \frac{pe^t}{1-qe^t} \quad \Longrightarrow \quad m_{X_1}(t) = m_{Y_1}\left(\frac{t}{n}\right) = \frac{pe^{t/n}}{1-qe^{t/n}}.$$

If $p = \lambda/n$, then

$$m_{X_1}(t) = \frac{(\lambda/n)e^{t/n}}{1-(1-\lambda/n)e^{t/n}} = \frac{\lambda e^{t/n}}{\lambda e^{t/n} + n(1-e^{t/n})} \longrightarrow \frac{\lambda}{\lambda - t}$$

as $n \longrightarrow \infty$, as $e^{t/n} \longrightarrow 1$ and

$$n(1-e^{t/n}) = n\left(1 - 1 - \frac{t}{n} - \frac{t^2}{2n^2} - \cdots\right) \longrightarrow -t.$$

# Connection to the Geometric distribution (cont.)

Suppose $X_1, X_2, X_3, \ldots, X_r, \ldots$ are defined in terms of the infinite sequence of independent $Bernoulli(p)$ trials and the random variables

$$Y_r = \text{“number of trials between } (r-1)\text{th and } r\text{th success”}$$

via the relationship

$$X_r = \frac{Y_r}{n}.$$

Then in the limit as $n \longrightarrow \infty$ with $\lambda = np$ fixed

$$X_1, X_2, X_3, \ldots, X_r, \ldots \sim Exponential(1/\lambda)$$

with these variables independent.

# Connection to the Poisson distribution

Let $Y \sim Gamma(r, 1/\lambda)$ for positive integer $r = 1, 2, \ldots$ and consider for $y \geq 0$ the cdf of $Y$,

$$F_Y(y) = P(Y \leq y) = \int_0^y \frac{\lambda^r}{(r-1)!} u^{r-1} e^{-\lambda u} \, du$$

with $F_Y(y) = 0$ for $y < 0$. Define

$$S_Y(r, y) = P(Y > y) = \int_y^\infty \frac{\lambda^r}{(r-1)!} u^{r-1} e^{-\lambda u} \, du$$

for $y > 0$.

# Connection to the Poisson distribution (cont.)

Integrating by parts, we have that

$$S_Y(r,y) = \left[-\frac{\lambda^{r-1}}{(r-1)!}u^{r-1}e^{-\lambda u}\right]_y^\infty + \int_y^\infty \frac{\lambda^{r-1}}{(r-2)!}u^{r-2}e^{-\lambda u}\,du$$

$$= \frac{\lambda^{r-1}}{(r-1)!}y^{r-1}e^{-\lambda y} + S_Y(r-1,y)$$

$$= \frac{\lambda^{r-1}}{(r-1)!}y^{r-1}e^{-\lambda y} + \frac{\lambda^{r-2}}{(r-2)!}y^{r-2}e^{-\lambda y} + S_Y(r-2,y)$$

using the recursion.

# Connection to the Poisson distribution (cont.)

By extending the recursion, we find that

$$S_Y(r, y) = \sum_{j=1}^{r-1} \frac{\lambda^j}{j!} y^j e^{-\lambda y} + S_Y(1, y).$$

We also have by direct calculation that

$$S_Y(1, y) = \int_y^\infty \lambda e^{-\lambda u} \, du = e^{-\lambda y}$$

so

$$S_Y(r, y) = \sum_{j=0}^{r-1} \frac{\lambda^j}{j!} y^j e^{-\lambda y}.$$

# Connection to the Poisson distribution (cont.)

Therefore, evaluating at $y = 1$

$$S_Y(r, 1) = \sum_{j=0}^{r-1} \frac{\lambda^j}{j!} e^{-\lambda}.$$

Thus we have that if $Y \sim Gamma(r, 1/\lambda)$,

$$P(Y > 1) = \sum_{j=0}^{r-1} \frac{\lambda^j}{j!} e^{-\lambda}.$$

# Connection to the Poisson distribution (cont.)

Recall the *Poisson process* definition of the Poisson distribution.

- The process $X(t)$ defines a collection of discrete random variables whose probabilistic properties are determined as a limiting case of a sequence of independent Bernoulli random variables.
- Events occur in continuous time at a constant rate $\lambda$, with events occurring independently;
- The number of events that occur in any interval of length $t$ is a random variable having a Poisson distribution with parameter $\lambda t$.

# Connection to the Poisson distribution (cont.)

The cdf of $X \equiv X(1)$ is therefore

$$F_X(x) = P(X \leq x) = \sum_{j=0}^{x} e^{-\lambda} \frac{\lambda^j}{j!}$$

for non-negative integer $x$. Hence

$$P(X \leq x-1) = \sum_{j=0}^{x-1} e^{-\lambda} \frac{\lambda^j}{j!} = P(Y > 1)$$

where $Y \sim Gamma(x, 1/\lambda)$.

# Connection to the Negative Binomial distribution

The negative binomial distribution arises in the context of a sequence of independent binary trials with success probability $p$ as the distribution of number of trials, $Y_r$, required to obtain the $r$th success.

The mgf of $Y_r \sim NegBinomial(r, p)$ takes the form

$$m_{Y_r}(t) = \left( \frac{pe^t}{1 - qe^t} \right)^r .$$

# Connection to the Negative Binomial distribution

We now let the number of trials $n$ tend to infinity, but fix $\lambda = np$. If we let $X_r = Y_r/n$ for fixed $r = 1, 2, 3, \ldots$, we have that

$$m_{X_r}(t) \longrightarrow m_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r$$

for $t < \lambda$. Thus the limiting mgf corresponds to the limiting random variable

$$X \sim Gamma(r, 1/\lambda).$$