MATH 323: PROBABILITY
MOMENT GENERATING FUNCTIONS

For discrete random variable Y with mass function p, the **moment-generating function** (mgf) of Y, $m(t)$, is defined by

$$m(t) = \sum_{j=0}^{\infty} \mu'_j \frac{t^j}{j!} = \mu'_0 + \mu'_1 t + \mu'_2 \frac{t^2}{2!} + \cdots$$

where $\mu'_j = \mathbb{E}[Y^j]$ is the jth moment for $j = 0, 1, 2, \ldots$. An equivalent representation is

$$m(t) = \mathbb{E}[e^{tY}] = \sum_y e^{ty} p(y)$$

provided this expectation exists for $t \in (-b, b)$ for some $b > 0$.

It can be shown that if Y_1 and Y_2 are random variables taking values on Y with pmf p_1 and p_2, and mgfs m_1 and m_2 respectively, then

$$p_1(y) \equiv p_2(y) \quad \forall y \in \mathcal{Y} \quad \iff \quad m_1(t) \equiv m_2(t) \quad \forall t \in (-b, b)$$

Hence there is a **1-1 correspondence between generating functions and distributions**: this provides a key technique for identification of probability distributions. Note that $m(0) = 1$, as

$$m(0) = \mathbb{E}[e^0] = \mathbb{E}[1] = \sum_y 1 \, p(y) = 1.$$

Also the kth derivative of m evaluated at t, $m^{(k)}(t)$, is given by

$$m^{(k)}(t) = \frac{d^k m(t)}{dt^k} = \frac{d^k}{dt^k} \left\{ \sum_{j=0}^{\infty} \mu'_j \frac{t^j}{j!} \right\}.$$

Hence

$$m^{(k)}(t) = \mu'_k + \mu'_{k+1} t + \mu'_{k+2} \frac{t^2}{2} + \cdots$$

so that

$$m^{(k)}(0) = \mu'_k = \mathbb{E}[Y^k].$$

Equivalently, we can obtain the same result for using the alternative representation.

$$m^{(k)}(t) = \frac{d^k m(t)}{dt^k} = \frac{d^k}{dt^k} \left\{ \sum_y e^{ty} p(y) \right\} = \sum_y y^k e^{ty} p(y)$$

and hence evaluating this function at $t = 0$,

$$m^{(k)}(0) = \sum_y y^k p(y) = \mathbb{E}[Y^k] = \mu'_k.$$

Suppose Y is a random variable whose distribution has mgf $m_Y(t)$. Suppose $X = aY + b$ is a new random variable defined in terms of Y. Then

$$\mathbb{E}[e^{tX}] \equiv \mathbb{E}[e^{t(aY+b)}] \quad \Longrightarrow \quad m_X(t) = \mathbb{E}[e^{t(aY+b)}] = e^{bt}\mathbb{E}[e^{(at)Y}] = e^{bt} m_Y(at).$$

Therefore $m_X(t) = e^{bt} m_Y(at)$.

The probability generating function

For random variable Y with pmf function p the **probability generating function** (pgf) of Y, G, is defined by

$$G(t) = \sum_y t^y p(y) = p(0) + tp(1) + t^2 p(2) + \cdots$$

whenever this sum is finite, so that

$$G(t) = \mathbb{E}[t^Y] = \mathbb{E}[e^{Y \log t}] = m(\log t)$$

if this expectation exists for all values of $t \in (1-b, 1+b)$ for some $b > 0$. From the definition, we see that the kth derivative of $G(t)$ with respect to t is

$$G^{(k)}(t) = \frac{d^k G(t)}{dt^k} = k!p(k) + \frac{(k+1)!}{2!} tp(k+1) + \cdots$$

so that evaluating at $t = 0$ removes all terms apart from the first, and hence

$$\frac{G^{(k)}(0)}{k!} = P(Y = k).$$

Also, using similar techniques to those used for the mgf, it can be shown that

$$G^{(k)}(t) = \frac{d^k G(t)}{dt^k} = \mathbb{E}\left[Y(Y-1)...(Y-k+1)t^{Y-k}\right]$$

so that

$$G^{(k)}(1) = \mathbb{E}[Y(Y-1)...(Y-k+1)]$$

where $\mathbb{E}[Y(Y-1)...(Y-k+1)]$ is the kth **factorial moment**.