MATH 323: PROBABILITY
AN APPLICATION OF CONDITIONAL PROBABILITY
THE MONTY HALL GAME SHOW PROBLEM

Problem: In a TV Game show, a contestant selects one of three doors; behind one of the doors there is a prize (in the common version of the problem, a car), and behind the other two there are no prizes (or in the common version, a goat behind each of these two doors). After the contestant selects a door, the game show host opens one of the remaining doors, and reveals that there is no prize behind it. The host then asks the contestant whether they want to SWITCH their choice to the other unopened door, or STICK to their original choice.

Is it probabilistically advantageous for the contestant to SWITCH doors, or is the probability of winning the prize the same whether they STICK or SWITCH ?

We may assume that the host selects a door to open, from those available to be opened, with equal probability.

Solution: Let S denote the sample space of outcomes corresponding to the experiment that records where the prize is concealed. Let events A, B, C correspond to the prize actually being behind the

- contestant's selected door (A)
- opened door (B), and
- remaining door (C)

respectively. Note that A, B and C form a partition of S. Let H_B denote the event that the host opens door B. Given H_B, we want to compare

$$P(A|H_B) \text{ (STICK)} \qquad \text{with} \qquad P(C|H_B) \text{ (SWITCH)}.$$

We have

$$P(A) = P(B) = P(C) = \frac{1}{3}$$

and also

$$P(H_B|A) = \frac{1}{2} \qquad P(H_B|B) = 0 \qquad P(H_B|C) = 1.$$

Therefore, as

$$H_B = (H_B \cap A) \cup (H_B \cap B) \cup (H_B \cap C)$$

by the partition, then

$$P(H_B) = P(H_B \cap A) + P(H_B \cap B) + P(H_B \cap C)$$

which we may rewrite

$$P(H_B) = P(H_B|A)P(A) + P(H_B|B)P(B) + P(H_B|C)P(C)$$

using the conditional probability definition. Then by Bayes theorem

$$\begin{aligned} P(A|H_B) &= \frac{P(H_B|A)P(A)}{P(H_B)} = \frac{P(H_B|A)P(A)}{P(H_B|A)P(A) + P(H_B|B)P(B) + P(H_B|C)P(C)} \\ &= \frac{\left(\frac{1}{2} \times \frac{1}{3}\right)}{\left(\frac{1}{2} \times \frac{1}{3}\right) + \left(0 \times \frac{1}{3}\right) + \left(1 \times \frac{1}{3}\right)} = \frac{1}{3} \end{aligned}$$

so $P(A|H_B) = 1/3$, and similarly $P(C|H_B) = 2/3$, **so it is advantageous to SWITCH**.