MATH 323: PROBABILITY
SOME USEFUL SUMMATION RESULTS

- **SERIES SUMMATIONS**

  GEOMETRIC: For $|x| < 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k$$

  EXPONENTIAL:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

  We also have by the definition of the exponential function that for real $x > 0$

$$\begin{aligned} e^x &= \lim_{n \longrightarrow \infty} \left(1 + \frac{x}{n}\right)^n &= \lim_{n \longrightarrow \infty} \left(1 - \frac{x}{n}\right)^{-n} \\ e^{-x} &= \lim_{n \longrightarrow \infty} \left(1 - \frac{x}{n}\right)^n &= \lim_{n \longrightarrow \infty} \left(1 + \frac{x}{n}\right)^{-n} \end{aligned}$$

  BINOMIAL: For $n > 0$,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + nx^{n-1} + x^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

- **TAYLOR SERIES:** For real function f and real number x_0

$$f(x) = \sum_{k=0}^{\infty} \frac{(x - x_0)^k}{k!} f^k(x_0)$$

  where

$$f^k(x_0) = \frac{d^k}{dx^k} \{f(x)\}_{x=x_0}$$

  assuming that the derivatives exist.

- **RESULTS RELATING TO THE GEOMETRIC SERIES:** We have from above that for $|x| < 1$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{k=0}^{\infty} x^k. \tag{1}$$

► **Differentiating** both sides of equation (1), we have

$$\frac{d}{dx}\left\{\frac{1}{1-x}\right\} = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{k=1}^{\infty} kx^{k-1}$$

$$\frac{d^2}{dx^2}\left\{\frac{1}{1-x}\right\} = \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots = \sum_{k=2}^{\infty} k(k-1)x^{k-2}$$

$$\frac{d^3}{dx^3}\left\{\frac{1}{1-x}\right\} = \frac{6}{(1-x)^4} = 6 + 24x + \cdots = \sum_{k=3}^{\infty} k(k-1)(k-2)x^{k-3}$$

If we differentiate n times, we therefore obtain

$$\frac{d^n}{dx^n}\left\{\frac{1}{1-x}\right\} = \frac{n!}{(1-x)^{n+1}} = \sum_{k=n}^{\infty} k(k-1)\ldots(k-n+1)x^{k-n}.$$

Re-writing the sum by re-setting the index $j = k - n$, we have

$$\frac{n!}{(1-x)^{n+1}} = \sum_{j=0}^{\infty}(j+n)(j+n-1)\ldots(j+1)x^j = \sum_{j=0}^{\infty}\frac{(j+n)!}{j!}x^j$$

Therefore we have the **NEGATIVE BINOMIAL** expansion: for $n > 0$ and $|x| < 1$

$$\frac{1}{(1-x)^{n+1}} = 1 + (n+1)\,x + \frac{(n+1)\,(n+2)}{2!}x^2 + \cdots = \sum_{k=0}^{\infty}\binom{n+k}{k}x^k$$

► **Integrating** the geometric series in equation (1), we obtain the following:

$$\int_0^x \frac{1}{1-t}\,dt = \int_0^x \left\{\sum_{k=0}^{\infty} t^k\right\}\,dt = \sum_{k=0}^{\infty}\left\{\int_0^x t^k\,dt\right\} = \sum_{k=0}^{\infty}\frac{x^{k+1}}{k+1} \equiv \sum_{k=1}^{\infty}\frac{x^k}{k}.$$

But by direct integration we have

$$\int_0^x \frac{1}{1-t}\,dt = [-\log(1-t)]_0^x = -\log(1-x)$$

that is

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots = \sum_{k=1}^{\infty}\frac{x^k}{k}$$

which yields the **LOGARITHMIC** series expansions; for $|x| < 1$,

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{k=1}^{\infty}\frac{x^k}{k}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{k=1}^{\infty}(-1)^{k+1}\,\frac{x^k}{k}$$

NB ***In the course,*** $\log$ ***denotes natural log and is identical to*** $\ln$**.**