MATH 323: Probability

December 20th, 2021
9.00am – 12.00pm.

First name (please write as legibly as possible within the boxes)

Last name

Student ID number

MATH 323: Specific Instructions.

Please write your answers on the exam in the boxes provided; only answers written inside the boxes will be marked. Please write clearly.

This paper contains five questions. Each question carries 20 marks. Credit will be given for all questions attempted. The total mark available is 100 but rescaling of the final mark may occur.

A formula sheet, a distribution sheet, a sheet of useful summation results, and a table of the standard Normal distribution are provided.

Only non-programmable calculators may be used. Dictionaries and Translation dictionaries are permitted.

The usual abbreviations are used: pmf indicates the probability mass function; pdf indicates the probability density function; cdf indicates the cumulative distribution function; mgf indicates the moment-generating function.

If the intention of a question is unclear to you, please state your interpretation and provide an answer accordingly. Your comment will be taken into account in marking.

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1. (a) For events A, B and C in sample space S, suppose that

$$P(A) = 0.4 \qquad P(B) = 0.3 \qquad P(C) = 0.1.$$

Write down

(i) an upper bound on $P(A \cup B \cup C)$, and 2 MARKS

(ii) an upper bound on $P(A \cap B \cap C)$ 2 MARKS

which is, in each case, less than 1. Justify your answer. If it is not possible to find a bound lower than 1, explain why.

Q1(a):

(b) Suppose that two events E and F partition S. Show that

$$P(E \cap F) + P(E \cup F) = 1.$$

2 MARKS

Q1(b):

(c) A soccer squad comprises 6 defenders, 6 midfielders, and 4 attackers as its outfield players. The team manager decides to select the outfield team with 4 defenders, 3 midfielders, and 3 attackers. How many possible teams with this configuration could be selected ? 4 MARKS

Q1(c):

(d) In a multiple choice test, each question has three possible answers: A,B,C. How many possible answer patterns are there, if it is known that each answer letter must appear precisely five times each ? 5 MARKS

Q1(d):

(e) A bag contains 10 balls: three are Red, three are Blue, three are Green and one is White. Balls are to be selected from the bag, with all balls equally likely to be selected from those that are still in the bag at every selection. What is the probability that

(i) the White ball is not selected on any of the first ten selections, if selections are made *with replacement* at each stage ?; 2 MARKS

(ii) two Red balls are selected during the first five selections, if selections are made *without replacement* at each stage ?. 3 MARKS

Q1(e):

2. (a) Two eggs are needed for a cake recipe, and they are selected from a box of twelve eggs. Unfortunately, the box contains two bad eggs that will affect the taste of the cake:

- if no bad eggs are selected, then the cake will have an acceptable taste with probability 0.95;
- if precisely one bad egg is selected, then the cake will have an acceptable taste with probability 0.60;
- if two bad eggs are selected, then the cake will have an acceptable taste with probability 0.10.

What is the probability that

(i) the cake does not have an acceptable taste ?; 5 MARKS

(ii) given that the cake does not have an acceptable taste, what is the probability that precisely one bad egg was used ? 5 MARKS

Assume that all selections of two eggs from twelve are equally likely.

Q2(a):

(b) A bag contains 10 balls which are known to be Red or White. Four balls are selected from the bag, with all such selections deemed to be equally likely.

(i) If the number of Red balls in the bag **known** to be five, compute the probability of obtaining three Red balls in the selection. 4 MARKS

(ii) If the number of Red balls in the bag is **unknown**, but it is believed *before* the selection that all possible numbers of Red balls in the bag are equally likely, derive an expression for the probability that number of Red balls is r, given that three of the balls in the sample are observed to be Red, for all suitable values of r.

6 MARKS

Q2(b):

3. (a) Suppose that a discrete random variable Y has mgf $m(t)$ given by

$$m(t) = c(1 + 2e^t + 3e^{2t})$$

for $t \in \mathbb{R}$ for some c.

(i) Find the value of c. 2 MARKS

(ii) Find $\mathbb{E}[Y^3]$. 3 MARKS

(iii) Find $\mathbb{E}[Y(Y-1)]$. 3 MARKS

(iv) Suppose that Y_1 and Y_2 are independent random variables having the same distribution as Y. Find the pmf of

$$X = Y_1 + Y_2.$$

6 MARKS

Q3(a):

(b) Suppose that Z_1 and Z_2 are independent Poisson random variables each with parameter λ. Let $S = Z_1 + Z_2$. Find the conditional probability that $Z_1 = z_1$, given that $S = s$ for s such that $P(S = s) > 0$. 6 MARKS

Hint: if $P(S = s) > 0$ *then*

$$P(Z_1 = z_1 | S = s) = \frac{P(Z_1 = z_1, S = s)}{P(S = s)}$$

for each relevant z_1*.*

Q3(b):

4. (a) For each of the following transformation problems, compute the distribution (that is, compute the pmf, pdf or cdf, or identify by name) of the variable U for the stated initial variable Y.

   (i) $U = 2Y$, if $Y \sim Geometric(p)$. 2 MARKS

   (ii) $U = 2Y$, if $Y \sim Exponential(1)$. 2 MARKS

   (iii) $U = \exp\{-Y/2\}$, if $Y \sim Exponential(2)$. 2 MARKS

   (iv) $U = (Y - \mu)^2/4$, if $Y \sim Normal(\mu, 2^2)$. 2 MARKS

   (v) $U = \Phi(Y)$, if $Y \sim Normal(0, 1)$, and $\Phi(.)$ is the standard Normal cdf. 2 MARKS

Q4(a):

(b) Suppose that Y_1 and Y_2 are independent random variables with $Y_1 \sim Exponential(\beta_1)$ and $Y_2 \sim Exponential(\beta_2)$. Find expressions for

(i) $P(Y_1 > 2|Y_1 > 1)$; 1 MARK

(ii) $P(Y_1 > 2|Y_2 > 1)$; 1 MARK

(iii) $P(Y_1 + Y_2 > 3)$, assuming that $\beta_1 = \beta_2 = \beta$, say; 6 MARKS

(iv) $P(Y_1 > 3, Y_2 > 2)$. 2 MARKS

Q4(b):

5. This question concerns sequences $Y_1, Y_2, \ldots, Y_n$ for $n \geq 1$ which are independent and identically distributed random variables. For any $n \geq 1$, define random variables

$$S_n = \sum_{i=1}^{n} Y_i \qquad \overline{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i.$$

(a) Suppose that $Y_1, Y_2, \ldots, Y_r$ each has a $Geometric(p)$ distribution, for parameter $0 < p < 1$, where $r \geq 2$ is a fixed value.

(i) Write down the expectation and variance of $\overline{Y}_r$. 2 MARKS

(ii) Find the distribution of S_r (that is, identify the distribution by name, or find the pmf). 4 MARKS

(iii) Find an expression for the conditional probability

$$P(Y_1 > 2 | Y_1 > 1).$$

2 MARKS

(iv) Find an expression for the conditional probability

$$P(Y_2 > 2 | Y_1 > 1)$$

2 MARKS

Q5(a):

Q5(a) (continued):

(b) Suppose that $Y_1, Y_2, \ldots, Y_n$ each has a $Normal(0, 1)$ distribution. Find the probability

$$P\left(\overline{Y}_n \leq \frac{1}{2}\right)$$

if $n = 4$. 4 MARKS

Q5(b):



(c) Suppose that $Y_1, Y_2, \ldots, Y_n$ each has a $Bernoulli(1/2)$ distribution. Using the Central Limit Theorem, find an approximation to the probability

$$P\left(\overline{Y}_n \leq \frac{1}{2}\right)$$

if n is large. 3 MARKS

Q5(c):

(d) Suppose that $Y_1, Y_2, \ldots, Y_n$ each has a $Normal(0, 1)$ distribution. For each i, define random variable

$$U_i = Y_i^2.$$

Describe the behaviour of

$$\overline{U}_n = \frac{1}{n} \sum_{i=1}^{n} U_i$$

as $n \longrightarrow \infty$. 3 MARKS

Q5(d):