MATH 323 Fall 2021: Formula Sheet

- **Basic Probability:** For events A and B in sample space S
  - *General Addition Rule:*
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
  - *de Morgan's Laws:*
$$P(A' \cap B') = P((A \cup B)') \qquad P(A' \cup B') = P((A \cap B)')$$

- **Combinatorics:**
  - *Multiplication principle:* A sequence of k operations, in which operation i can result in n_i possible outcomes, can result in $n_1 \times n_2 \times \cdots \times n_k$ possible sequences of outcomes.
  - *Permutations:* An ordered arrangement of r distinct objects is called a *permutation*. The number of ways of ordering r distinct objects taken from n without replacement is denoted by
$$P_r^n = \frac{n!}{(n-r)!}$$
  - *Multinomial coefficients:* The number of ways of partitioning n distinct objects into k disjoint subsets of sizes $n_1, n_2, \ldots, n_k$ where $n_1 + n_2 + \cdots + n_k = n$ is
$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! \times n_2! \ldots \times n_k!}.$$
  - *Combinations:* The number of *combinations* of n objects taken r at a time is the number of subsets of size r that can be formed. This number is denoted
$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$
  - *Binary sequences:* the number of binary sequences of length n containing r 1s is C_r^n.
  - *Hypergeometric probabilities:* Finite population of size N comprises R Type I objects and $N - R$ Type II objects, and a sample of size $n \leq N$ is obtained from the population. The probability that the sample contains r Type I objects and $n - r$ Type II objects is
$$\frac{\binom{n}{r}\binom{N-n}{R-r}}{\binom{N}{R}} = \frac{\binom{R}{r}\binom{N-R}{n-r}}{\binom{N}{n}}$$

- **Probability Theorems:** Suppose events $B_1, B_2, \ldots, B_n$ form a partition of sample space S, and let $A \subseteq S$. Provided all the conditional probabilities are well-defined, we have
  - *Theorem of Total Probability:*
$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i).$$
  - *Bayes Theorem:*
$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A)} = \frac{P(A|B_i)P(B_i)}{\sum_{j=1}^{n} P(A|B_j)P(B_j)}.$$

- **Probability Distributions & Properties:**
  - *Distributions:*
    - *Discrete case:* A discrete probability distribution places probability a countable number of real values. For $y \in \mathbb{R}$, we the pmf is the function

      $$p(y) = P(Y = y)$$

      The cdf is the function

      $$F(y) = P(Y \leq y)$$

      In the discrete case, $F(y)$ is a step function.
    - *Continuous case:* A continuous probability distribution is a distribution where the cdf

      $$F(y) = P(Y \leq y)$$

      is continuous for all $y \in \mathbb{R}$. The pdf, $f(.)$, is the function for which

      $$F(y) = \int_{-\infty}^{y} f(t)\, dt$$

      so that

      $$f(y) = \frac{dF(y)}{dy}$$

      wherever the derivative exists.
  - *Expectations:* The expectation of a function $g(Y)$ of random variable Y is defined by

    $$\mathbb{E}[g(Y)] = \begin{cases} \sum\limits_{y} g(y)p(y) & \text{Discrete case} \\ \int_{-\infty}^{\infty} g(y)f(y)\, dy & \text{Continuous case} \end{cases}$$

    whenever the sum or integral is well-defined.
  - *mgfs:* The mgf of random variable Y, $m(t)$, is defined by

    $$m(t) = \sum_{j=0}^{\infty} \mu'_j \frac{t^j}{j!} = \mu'_0 + \mu'_1 t + \mu'_2 \frac{t^2}{2!} + \cdots$$

    where $\mu'_j = \mathbb{E}[Y^j]$ is the jth moment for $j = 0, 1, 2, \ldots$. We have

    $$m(t) = \mathbb{E}[e^{tY}] = \begin{cases} \sum\limits_{y} e^{ty}p(y) & \text{Discrete case} \\ \int_{-\infty}^{\infty} e^{ty}f(y)\, dy & \text{Continuous case} \end{cases}$$

    provided the expectation exists for $t \in (-b, b)$ for some $b > 0$.
    - $m(0) = 1$ and
    - For $k = 1, 2, \ldots$,

      $$m^{(k)}(0) = \left.\frac{d^k m(t)}{dt^k}\right|_{t=0} = \mathbb{E}[Y^k]$$

    - If Y has mgf $m_Y(t)$, and $X = \mu + \sigma Y$, then $m_X(t) = e^{\mu t} m_Y(\sigma t)$.

- ▶ **Sums of independent random variables:**
  - ∗ *Convolution result:* If Y_1 and Y_2 are independent random variables, and $Y = Y_1 + Y_2$, we have that the distribution of Y is given by

$$p_Y(y) = \sum_{y_1=-\infty}^{\infty} p_{Y_1}(y_1) p_{Y_2}(y - y_1) \qquad \text{Discrete case}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y_1}(y_1) f_{Y_2}(y - y_1)\, dy_1 \qquad \text{Continuous case}$$

  - ∗ *mgfs:* If $Y_1, Y_2, \ldots, Y_n$ are independent random variables with mgfs $m_1(t), \ldots, m_n(t)$, and

$$Y = \sum_{i=1}^{n} Y_i$$

  then the mgf of Y, $m_Y(t)$, is computed as

$$m_Y(t) = m_1(t) \times m_2(t) \times \ldots \times m_n(t) = \prod_{i=1}^{n} m_i(t).$$

- **Inequalities:**
  - ▶ *Markov's inequality* If $g(Y)$ is a function of random variable Y such that $\mathbb{E}[|g(Y)|] < \infty$. For any $k > 0$,

$$P(|g(Y)| \geq k) \leq \frac{\mathbb{E}[|g(Y)|]}{k} \qquad \text{or} \qquad P(|g(Y)| < k) \geq 1 - \frac{\mathbb{E}[|g(Y)|]}{k}.$$

  - ▶ *Chebychev's Inequality* Suppose that Y has finite mean μ and variance σ^2.

$$P(|Y - \mu| < c\sigma) \geq 1 - \frac{1}{c^2} \qquad \text{or} \qquad P(|Y - \mu| \geq c\sigma) \leq \frac{1}{c^2}.$$

- **Limit Theorems:** Suppose that $Y_1, Y_2, \ldots, Y_n$, are independent random variables that have the same distribution, and have expectation μ and variance σ^2.
  - ▶ *The Weak Law of Large Numbers (WLLN):* For each n, let

$$\overline{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i$$

  be the sample mean random variable. Then $\overline{Y}_n \xrightarrow{p} \mu$ as $n \longrightarrow \infty$, that is, the probability distribution of $\overline{Y}_n$ becomes **concentrated** at μ.
  - ▶ *The Central Limit Theorem (CLT):* The random variable

$$U_n = \frac{\sqrt{n}(\overline{Y}_n - \mu)}{\sigma}$$

  converges in distribution to a *standard Normal distribution*. When n is large, we can approximate the distributions of $\overline{Y}_n$, or of

$$S_n = \sum_{i=1}^{n} Y_i$$

  using a Normal distribution, irrespective of the actual distribution of $Y_1, \ldots, Y_n$. That is
    - ∗ $\overline{Y}_n$ is approximately distributed as

$$Normal(\mu, \sigma^2/n)$$

    - ∗ S_n is approximately distributed as

$$Normal(n\mu, n\sigma^2)$$

DISCRETE DISTRIBUTIONS

	$\mathcal{Y}$	PARAMETERS	$p(y)$	$F(y)$	$\mathbb{E}[Y]$	$\mathbb{V}[Y]$	$m(t)$
$Bernoulli(p)$	$\{0,1\}$	$p \in (0,1)$	$p^y(1-p)^{1-y}$		p	$p(1-p)$	$1-p+pe^t$
$Binomial(n,p)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, p \in (0,1)$	$\binom{n}{y}p^y(1-p)^{n-y}$		np	$np(1-p)$	$(1-p+pe^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\frac{e^{-\lambda}\lambda^y}{y!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(p)$	$\{1,2,...\}$	$p \in (0,1)$	$(1-p)^{y-1}p$	$1-(1-p)^y$	$\frac{1}{p}$	$\frac{(1-p)}{p^2}$	$\frac{pe^t}{1-e^t(1-p)}$
$NegBinomial(r,p)$	$\{r,r+1,...\}$	$r \in \mathbb{Z}^+, p \in (0,1)$	$\binom{y-1}{r-1}p^r(1-p)^{y-r}$		$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left(\frac{pe^t}{1-e^t(1-p)}\right)^r$
or	$\{0,1,2,...\}$	$r \in \mathbb{Z}^+, p \in (0,1)$	$\binom{r+y-1}{r-1}p^r(1-p)^y$		$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$	$\left(\frac{p}{1-e^t(1-p)}\right)^r$
$Hypergeom(N,r,n)$	$\{\max\{0,n-N+r\},\ldots$ $\ldots,\min\{n,r\}\}$	$N \geq r,n$	$\frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}$		$n\left(\frac{r}{N}\right)$	$n\left(\frac{r}{N}\right)\left(1-\frac{r}{N}\right)\left(\frac{N-n}{N-1}\right)$	
or			$\frac{\binom{n}{y}\binom{N-n}{r-y}}{\binom{N}{r}}$				

CONTINUOUS DISTRIBUTIONS							
	$\mathcal{Y}$	PARAMETERS	$f(y)$	$F(y)$	$\mathbb{E}[Y]$	$\mathbb{V}[Y]$	$m(t)$
$Uniform(\theta_1, \theta_2)$ (standard model $\theta_1 = 0, \theta_2 = 1$)	(θ_1, θ_2)	$\theta_1 < \theta_2 \in \mathbb{R}$	$\frac{1}{\theta_2 - \theta_1}$	$\frac{y - \theta_1}{\theta_2 - \theta_1}$	$\frac{(\theta_1 + \theta_2)}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{\theta_2 t} - e^{\theta_1 t}}{t(\theta_2 - \theta_1)}$
$Exponential(\beta)$ (standard model $\beta = 1$)	$\mathbb{R}^+$	$\beta \in \mathbb{R}^+$	$\frac{1}{\beta} e^{-y/\beta}$	$1 - e^{-y/\beta}$	β	β^2	$\left(\frac{1}{1 - \beta t}\right)$
$Gamma(\alpha, \beta)$ (standard model $\beta = 1$)	$\mathbb{R}^+$	$\alpha, \beta \in \mathbb{R}^+$	$\frac{1}{\beta^\alpha \Gamma(\alpha)} y^{\alpha-1} e^{-y/\beta}$ where $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy$		$\alpha\beta$	$\alpha\beta^2$	$\left(\frac{1}{1 - \beta t}\right)^\alpha$
$Normal(\mu, \sigma^2)$ (standard model $\mu = 0, \sigma = 1$)	$\mathbb{R}$	$\mu \in \mathbb{R}, \sigma \in \mathbb{R}^+$	$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$\exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$
$Beta(\alpha, \beta)$	$(0, 1)$	$\alpha, \beta \in \mathbb{R}^+$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}$		$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
$Weibull(\alpha, \beta)$ (standard model $\beta = 1$)	$\mathbb{R}^+$	$\alpha, \beta \in \mathbb{R}^+$	$\alpha\beta y^{\alpha-1} e^{-\beta y^\alpha}$	$1 - e^{-\beta y^\alpha}$	$\frac{\Gamma(1 + 1/\alpha)}{\beta^{1/\alpha}}$	$\frac{\Gamma(1 + 2/\alpha) - \Gamma(1 + 1/\alpha)^2}{\beta^{2/\alpha}}$	
$Student(\nu)$	$\mathbb{R}$	$\nu \in \mathbb{R}^+$	$\frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)\sqrt{\pi\nu}\{1 + y^2/\nu\}^{(\nu+1)/2}}$		0 (if $\nu > 1$)	$\frac{\nu}{\nu - 2}$ (if $\nu > 2$)	
$Pareto(\theta, \alpha)$	$\mathbb{R}^+$	$\theta, \alpha \in \mathbb{R}^+$	$\frac{\alpha\theta^\alpha}{(\theta + y)^{\alpha+1}}$	$1 - \left(\frac{\theta}{\theta + y}\right)^\alpha$	$\frac{\theta}{\alpha - 1}$ (if $\alpha > 1$)	$\frac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha > 2$)	

For linear transformation $X = \mu + \sigma Y$

$$f_X(x) = f_Y\left(\frac{x-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_X(x) = F_Y\left(\frac{x-\mu}{\sigma}\right) \qquad m_X(t) = e^{\mu t} m_Y(\sigma t) \qquad \mathbb{E}[X] = \mu + \sigma\mathbb{E}[Y] \qquad \mathbb{V}[X] = \sigma^2\mathbb{V}[Y]$$

Useful Series Summations

GEOMETRIC: For $|t| < 1$,

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots = \sum_{k=0}^{\infty} t^k$$

For the finite series, for $n \geq 1$

$$\frac{1-t^{n+1}}{1-t} = 1 + t + \cdots + t^n$$

EXPONENTIAL:

$$e^t = 1 + t + \frac{t^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{t^k}{k!}$$

$$e^t = \lim_{n \longrightarrow \infty} \left(1 + \frac{t}{n}\right)^n = \lim_{n \longrightarrow \infty} \left(1 - \frac{t}{n}\right)^{-n}$$

$$e^{-t} = \lim_{n \longrightarrow \infty} \left(1 - \frac{t}{n}\right)^n = \lim_{n \longrightarrow \infty} \left(1 + \frac{t}{n}\right)^{-n}$$

BINOMIAL: For $n > 0$,

$$(1+t)^n = 1 + nt + \frac{n(n-1)}{2!}t^2 + \cdots + nt^{n-1} + t^n = \sum_{k=0}^{n} \binom{n}{k} t^k$$

NEGATIVE BINOMIAL: for $n > 0$ and $|t| < 1$,

$$\frac{1}{(1-t)^{n+1}} = 1 + (n+1)\,t + \frac{(n+1)\,(n+2)}{2!}t^2 + \cdots = \sum_{k=0}^{\infty} \binom{n+k}{k} t^k$$

LOGARITHMIC: for $|t| < 1$,

$$-\log(1-t) = t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots = \sum_{k=1}^{\infty} \frac{t^k}{k}$$

$$\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{t^k}{k}$$

Table of the standard Normal distribution

Entries in table are $P(Y \leq y)$

y	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990