MATH 323 - EXERCISES 9- SOLUTIONS

1. **Binomial:**

   Convolution: For $y = 0, 1, ..., n_1 + n_2$,

$$\begin{aligned} p_Y(y) &= \sum_{y_1=-\infty}^{\infty} p_{Y_1}(y_1) p_{Y_2}(y - y_1) \\ &= \sum_{y_1=0}^{y} \binom{n_1}{y_1} p_1^y (1-p)^{n_1 - y_1} \binom{n_2}{y - y_1} p^{t - y_1} (1-p)^{n_2 - y + y_1} \\ &= \sum_{y_1=0}^{y} \binom{n_1}{y_1} \binom{n_2}{y - y_1} p^y (1-p)^{n_1 + n_2 - y} \\ &= p^y (1-p)^{n_1+n_2-y} \sum_{y_1=0}^{y} \binom{n_1}{y_1} \binom{n_2}{y - y_1} \\ &= \binom{n_1 + n_2}{y} p^y (1-p)^{n_1+n_2-y} \end{aligned}$$

   where the last line follows from the hypergeometric identity established earlier

$$\binom{n_1 + n_2}{y} = \sum_{y_1=0}^{y} \binom{n_1}{y_1} \binom{n_2}{y - y_1}.$$

   Hence $Y \sim Binomial(n_1 + n_2, p)$.

   Mgfs: from the formula sheet

$$m_{Y_1}(t) = (1 - p + pe^t)^{n_1} \qquad m_{Y_2}(t) = (1 - p + pe^t)^{n_2}$$

   so by the stated result

$$m_Y(t) = (1 - p + pe^t)^{n_1} (1 - p + pe^t)^{n_2} = (1 - p + pe^t)^{n_1 + n_2}.$$

   Hence $Y \sim Binomial(n_1 + n_2, p)$.

2. **Poisson:**

   Convolution: For $y = 0, 1, 2, \ldots$,

$$\begin{aligned} p_Y(y) &= \sum_{y_1=-\infty}^{\infty} p_{Y_1}(y_1) p_{Y_2}(y - y_1) = \sum_{y_1=0}^{y} \frac{{\lambda_1}^{y}}{y_1!} e^{-\lambda_1} \frac{{\lambda_2}^{y - y_1}}{(y - y_1)!} e^{-\lambda_2} \\ &= \frac{(\lambda_1 + \lambda_2)^y}{y!} e^{-(\lambda_1 + \lambda_2)} \left[\sum_{y_1=0}^{y} \binom{y}{y_1} \left\{ \frac{\lambda_1}{\lambda_1 + \lambda_2} \right\}^{y_1} \left\{ \frac{\lambda_2}{\lambda_1 + \lambda_2} \right\}^{y - y_1} \right] \\ &= \frac{(\lambda_1 + \lambda_2)^y}{y!} e^{-(\lambda_1 + \lambda_2)} \end{aligned}$$

where the last line follows as the sum is a binomial expansion

$$\sum_{y_1=0}^{y} \binom{y}{y_1} \left\{\frac{\lambda_1}{\lambda_1+\lambda_2}\right\}^{y_1} \left\{\frac{\lambda_2}{\lambda_1+\lambda_2}\right\}^{y-y_1} = \left(\frac{\lambda_1}{\lambda_1+\lambda_2}+\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{y} = 1.$$

Hence $Y \sim Poisson(\lambda_1+\lambda_2)$.

Mgfs: from the formula sheet

$$m_{Y_1}(t) = \exp\{\lambda_1(e^t-1)\} \qquad m_{Y_2}(t) = \exp\{\lambda_2(e^t-1)\}$$

so by the stated result

$$m_Y(t) = \exp\{\lambda_1(e^t-1)\}\exp\{\lambda_2(e^t-1)\} = \exp\{(\lambda_1+\lambda_2)(e^t-1)\}$$

Hence $Y \sim Poisson(\lambda_1+\lambda_2)$.

3. **Geometric:**

Convolution: As per the previous Assignment solution, for $y = 0, 1, 2, \ldots$,

$$\begin{aligned} p_Y(y) &= \sum_{y_1=-\infty}^{\infty} p_{Y_1}(y_1)p_{Y_2}(y-y_1) \\ &= \sum_{y_1=1}^{y-1} (1-p)^{y_1-1}p\,(1-p)^{y-y_1-1}p \\ &= p^2(1-p)^{y-2}\sum_{y_1=1}^{y-1} 1 \\ &= (y-1)p^2(1-p)^{y-2} \end{aligned}$$

Hence $Y \sim NegativeBinomial(2,p)$.

Mgfs: from the formula sheet

$$m_{Y_1}(t) = \left(\frac{pe^t}{1-e^t(1-p)}\right) \qquad m_{Y_2}(t) = \left(\frac{pe^t}{1-e^t(1-p)}\right)$$

so by the stated result

$$m_Y(t) = \left(\frac{pe^t}{1-e^t(1-p)}\right)^2$$

Hence $Y \sim NegativeBinomial(2,p)$.

4. **Exponential:**

Convolution: For $y > 0$, letting $\lambda = 1/\beta$,

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{Y_1}(y_1)f_{Y_2}(y-y_1)\,dy_1 \\ &= \int_0^y \lambda e^{-\lambda y_1}\lambda e^{-\lambda(y-y_1)}\,dy_1 \\ &= \lambda^2 e^{-\lambda y}\int_0^y dy_1 = \lambda^2 y e^{-\lambda y} \end{aligned}$$

Hence $Y \sim Gamma(2, \beta)$.

Mgfs: from the formula sheet

$$m_{Y_1}(t) = \left(\frac{1}{1-\beta t}\right) \qquad m_{Y_2}(t) = \left(\frac{1}{1-\beta t}\right)$$

so by the stated result

$$m_Y(t) = \left(\frac{1}{1-\beta t}\right)^2$$

Hence $Y \sim Gamma(2, \beta)$.

5. **Gamma:**

Convolution: For $y > 0$, letting $\lambda = 1/\beta$,

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{Y_1}(y_1) f_{Y_2}(y - y_1)\, dy_1 \\
&= \int_0^y \frac{\lambda^{\alpha_1}}{\Gamma(\alpha_1)} y_1^{\alpha_1 - 1} e^{-\lambda y_1} \frac{\lambda^{\alpha_2}}{\Gamma(\alpha_2)} (y - y_1)^{\alpha_2 - 1} \lambda e^{-\lambda(y - y_1)}\, dy_1 \\
&= \frac{\lambda^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\lambda y} \int_0^y y_1^{\alpha_1 - 1} (y - y_1)^{\alpha_2 - 1}\, dy_1 \\
&= \frac{\lambda^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\lambda y} y^{\alpha_1 + \alpha_2 - 1} \int_0^1 u^{\alpha_1 - 1} (1 - u)^{\alpha_2 - 1} du \qquad \text{setting } u = y_1/y \\
&= \frac{\lambda^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1 + \alpha_2)} y^{\alpha_1 + \alpha_2 - 1} e^{-\lambda y}
\end{aligned}$$

as the integrand is proportional to a Beta pdf

$$\int_0^1 u^{\alpha_1 - 1} (1 - u)^{\alpha_2 - 1} = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}.$$

Hence $Y \sim Gamma(\alpha_1 + \alpha_2, \beta)$.

Mgfs: from the formula sheet

$$m_{Y_1}(t) = \left(\frac{1}{1-\beta t}\right)^{\alpha_1} \qquad m_{Y_2}(t) = \left(\frac{1}{1-\beta t}\right)^{\alpha_2}$$

so by the stated result

$$m_Y(t) = \left(\frac{1}{1-\beta t}\right)^{\alpha_1 + \alpha_2}$$

Hence $Y \sim Gamma(\alpha_1 + \alpha_2, \beta)$.

6. **Normal:**

Convolution: For $y \in \mathbb{R}$

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{Y_1}(y_1) f_{Y_2}(y - y_1)\, dy_1 \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi\sigma_1^2}\right)^{1/2} \exp\left\{-\frac{1}{2\sigma_1^2}(y_1 - \mu_1)^2\right\} \left(\frac{1}{2\pi\sigma_2^2}\right)^{1/2} \exp\left\{-\frac{1}{2\sigma_2^2}(y - y_1 - \mu_Y)^2\right\} dy_1 \\
&= \left(\frac{1}{2\pi\sigma_1^2}\right)^{1/2} \left(\frac{1}{2\pi\sigma_2^2}\right)^{1/2} \exp\left\{-\frac{1}{2\sigma^2}(y - \mu)^2\right\} \int_{-\infty}^{\infty} \exp\left\{-\frac{\sigma^2}{2\sigma_1^2\sigma_2^2}\left(y - \frac{\sigma_2\mu - \sigma_2^2 y_1}{\sigma^2}\right)^2\right\} dy_1 \\
&= \left(\frac{1}{2\pi\sigma_1^2}\right)^{1/2} \left(\frac{1}{2\pi\sigma_2^2}\right)^{1/2} \exp\left\{-\frac{1}{2\sigma^2}(y - \mu)^2\right\} \sqrt{2\pi\frac{\sigma_1^2\sigma_2^2}{\sigma^2}} \\
&= \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left\{-\frac{1}{2\sigma^2}(y - \mu)^2\right\}
\end{aligned}$$

where $\mu = \mu_1 + \mu_2$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2$. The third line follows from the second by completing the square in y_1 in the exponential term:

$$\frac{1}{\sigma_1^2}(y_1 - \mu_1)^2 + \frac{1}{\sigma_2^2}(y - y_1 - \mu_2)^2 = \frac{\sigma^2}{\sigma_1^2\sigma_2^2}\left(y - \frac{\sigma_2\mu - \sigma_2^2 y_1}{\sigma^2}\right)^2 + \frac{1}{\sigma^2}(y_1 - \mu)^2$$

The fourth line follows from the third as the integrand is proportional to a Normal pdf with mean

$$\frac{\sigma_2\mu - \sigma_2^2 y_1}{\sigma^2}$$

and variance

$$\frac{\sigma_1^2\sigma_2^2}{\sigma^2}$$

Hence $Y \sim Normal(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Mgfs: from the formula sheet

$$m_{Y_1}(t) = \exp\left\{\mu_1 t + \frac{\sigma_1^2 t^2}{2}\right\} \qquad m_{Y_2}(t) = \exp\left\{\mu_2 t + \frac{\sigma_2^2 t^2}{2}\right\}$$

so by the stated result

$$m_Y(t) = \exp\left\{(\mu_1 + \mu_2)t + \frac{(\sigma_1^2 + \sigma_2^2)t^2}{2}\right\}$$

Hence $Y \sim Normal(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.