MATH 323 - EXERCISES 8- SOLUTIONS

1. From first principles

   - $U_1 = 2Y - 1$, so $\mathcal{U}_1 = [-1, 1]$

     $$F_{U_1}(u) = P(U_1 \le u) = P(2Y - 1 \le u) = P(Y \le (1+u)/2) = F_Y((1+u)/2)$$

     so

     $$f_{U_1}(u) = \frac{1}{2} f_Y((1+u)/2) = (1-u)/2 \quad -1 \le u \le 1$$

     and zero otherwise.

   - $U_2 = 1 - 2Y$, so $\mathcal{U}_2 = [-1, 1]$

     $$F_{U_2}(u) = P(U_2 \le u) = P(1 - 2Y \le u) = P(Y \ge (1-u)/2) = 1 - F_Y((1-u)/2)$$

     so

     $$f_{U_2}(u) = \frac{1}{2} f_Y((1-u)/2) = (1+u)/2 \quad -1 \le u \le 1$$

     and zero otherwise.

   - $U_3 = Y^2$, so $\mathcal{U}_3 = [0, 1]$

     $$F_{U_3}(u) = P(U_3 \le u) = P(Y^2 \le u) = P(-\sqrt{u} \le Y \le \sqrt{u}) = F_Y(\sqrt{u}) - F_Y(-\sqrt{u})$$

     so

     $$f_{U_3}(u) = \frac{1}{2\sqrt{u}} \left(f_Y(\sqrt{u}) + f_Y(-\sqrt{u})\right) = \frac{1}{2\sqrt{u}} f_Y(\sqrt{u}\ ) = (1 - \sqrt{u})/\sqrt{u} \quad 0 \le u \le 1$$

     and zero otherwise, as $f_Y(y) = 0$ for $y < 0$.

   We could also use the general transformation formula for the first two calculations, as the transformations are 1-1

   $$g(t) = 2t - 1 \Longleftrightarrow g^{-1}(t) = (1+t)/2 \qquad\qquad g(t) = 1 - 2t \Longleftrightarrow g^{-1}(t) = (1-t)/2$$

   then use

   $$f_U(u) = f_Y(g^{-1}(u)) \left| \frac{d}{du} \left\{ g^{-1}(u) \right\} \right|.$$

2. Using the general result for expectations of functions with $g(Y) = 2\left(1 - e^{-2Y}\right)$, with $\lambda = 1/\beta$,

   $$\begin{aligned}
   \mathbb{E}[g(Y)] &= \int_{-\infty}^{\infty} g(y) f_Y(y)\, dy = \int_0^{\infty} 2\left(1 - e^{-2y}\right) \lambda e^{-\lambda y}\, dy \\
   &= 2\lambda \int_0^{\infty} \left(e^{-\lambda y} - e^{-(2+\lambda)y}\right) dy \\
   &= 2\lambda \left(\frac{1}{\lambda} - \frac{1}{2+\lambda}\right) = \frac{4}{2+\lambda} = \frac{1}{2}
   \end{aligned}$$

3. $f_Y(y) = 1/2$, for $1 \leq y \leq 1$ and zero otherwise. From first principles, $U = |Y|$ so $\mathcal{U} = [0,1]$

$$F_U(u) = P(U \leq u) = P(|Y| \leq u) = P(-u \leq Y \leq u) = F_Y(y) - F_Y(-y)$$

so

$$f_U(u) = f_Y(u) + f_Y(-u) = 1 \qquad 0 \leq u \leq 1$$

and zero otherwise. Also $Z = Y^2$, so $\mathcal{Z} = [0,1]$, and

$$F_Z(z) = P(Z \leq z) = P(X^2 \leq z) = P(-\sqrt{z} \leq X \leq \sqrt{z}) = F_X(\sqrt{z}) - F_X(-\sqrt{z})$$

so therefore

$$f_Z(z) = \frac{1}{2\sqrt{z}} \left(f_X(\sqrt{z}) + f_X(-\sqrt{z})\right) = \frac{1}{2\sqrt{z}} \quad 0 \leq z \leq 1$$

and zero otherwise.

4. From first principles, if Y has cdf F_Y, then as Y is continuous, F_Y is 1-1 and monotone increasing. If $U = F_Y(Y)$ then $\mathcal{U} = [0,1]$

$$F_U(u) = P(U \leq u) = P(F_Y(Y) \leq u) = P[Y \leq F_Y^{-1}(u)) = F_Y(F_Y^{-1}(u)) = u \qquad 0 \leq u \leq 1.$$

so $U \sim Uniform(0,1)$. Next $Z = -\ln F_X(X) = -\ln U$, then $\mathcal{Z} = \mathbb{R}^+$, so

$$F_Z(z) = P(Z \leq z) = P(-\ln U \leq z) = P(U \geq e^{-z}) = 1 - F_U(e^{-z}) = 1 - e^{-z} \quad z > 0$$

so $Y \sim Exponential(1)$.

5. From first principles, if Y has a Weibull distribution, then $U = Y^\alpha$, so $\mathcal{U} = \mathbb{R}^+$

$$F_U(u) = P(U \leq y) = P(Y^\alpha \leq u) = P(Y \leq u^{1/\alpha}) = F_Y(u^{1/\alpha}) = 1 - e^{-\beta u} \qquad u > 0$$

so $U \sim Exponential(1/\beta)$.

6. $Y \sim Normal(0,1)$, and thus if Φ and ϕ are the standard normal cdf and pdf respectively, we have immediately that $X = Y^2$ implies $\mathcal{X} = \mathbb{R}^+$.

   (a) We have

   $$F_X(x) = P[X \leq x) = P[\,Y^2 \leq x\,] = P(-\sqrt{x} \leq Y \leq \sqrt{x}\,] = \Phi(\sqrt{x}) - \Phi(-\sqrt{x}).$$

   (b) By differentiation, we have

   $$f_X(x) = \frac{1}{2\sqrt{x}} \left(\phi(\sqrt{x}) + \phi(\sqrt{x})\right) = \left(\frac{1}{2\pi}\right)^{1/2} x^{-1/2}\, e^{-x/2} \quad 0 \leq x \leq \infty$$

   (c) By inspection, we have that $X \sim Gamma\,(1/2, 2) \equiv \chi^2_1$

7. **Convolution:** We may use the convolution result directly based on the result from Q6 (b). If $X_1 = Y_1^2$ and $X_2 = Y_2^2$, then we have that $X_1 \sim Gamma(1/2,2)$ and $X_2 \sim Gamma(1/2,2)$ with X_1 and X_2 independent. Then by the convolution result

$$f_V(v) = \int_0^v f_{X_1}(x_1) f_{X_2}(v - x_1)\, dx_1$$

as both X_1 and X_2 are non-negative. We have from Q6(b)

$$\begin{aligned}
f_V(v) &= \int_0^v \left(\frac{1}{2\pi}\right)^{1/2} x_1^{-1/2}\, e^{-x_1/2} \left(\frac{1}{2\pi}\right)^{1/2} (v-x_1)^{-1/2}\, e^{-(v-x_1)/2}\, dx_1 \\
&= \frac{1}{2\pi} e^{-v/2} \int_0^v x_1^{-1/2} (v-x_1)^{-1/2}\, dx_1 \\
&= \frac{1}{2\pi} e^{-v/2} \int_0^1 t^{-1/2}(1-t)^{-1/2}\, dt \qquad (t = x_1/v).
\end{aligned}$$

The integral does not depend on v, so the pdf is proportional to $e^{-v/2}$ for $v > 0$. Therefore we can conclude that $V \sim Exponential(2)$. Note that we have from this result that for the Beta function

$$B(1/2, 1/2) = \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)} = \int_0^1 t^{-1/2}(1-t)^{-1/2} = \pi$$

and as $\Gamma(1) = 1$, we deduce that $\Gamma(1/2) = \sqrt{\pi}$

From first principles: By using a joint pdf approach, we could also write

$$F_V(v) = P\left(V \le v\right) = P\left(Y_1^2 + Y_2^2 \le v\right) = \iint_{A_v} f_{Y_1}(y_1) f_{Y_2}(y_2)\, dy_1 dy_2$$

where $A_v = \left\{(y_1, y_2) \,:\, y_1^2 + y_2^2 \le v\right\}$ that is, the integral over the region A_v of the joint density function of Y_1 and Y_2. Now, should reparameterize into polar coordinates in the double integral; let $y_1 = r\cos\theta$ and $y_2 = r\sin\theta$. Then

$$\begin{aligned}
F_V(v) = \iint_{A_v} f_{Y_1}(y_1) f_{Y_2}(y_2)\, dy_1 dy_2 &= \iint_{A_v} \frac{1}{2\pi} e^{-(y_1^2+y_2^2)/2}\, dy_1 dy_2 \\
&= \int_{r=0}^{\sqrt{v}} \int_0^{2\pi} \frac{1}{2\pi} e^{-r^2/2} r\, d\theta dr \\
&= \int_{r=0}^{\sqrt{v}} r e^{-r^2/2}\, dr = 1 - e^{-v/2} \qquad z > 0
\end{aligned}$$

so $V \sim Exponential(2) \equiv \chi_2^2$.

Using mgfs: If $Y \sim Normal(0,1)$ then $X = Y^2 \sim Gamma(1/2, 2)$ and hence

$$m_X(t) = \int_0^\infty e^{ty} \left(\frac{1}{2\pi}\right)^{1/2} y^{-1/2}\, e^{-y/2}\, dy = \left(\frac{1}{2\pi}\right)^{1/2} \int_0^\infty y^{-1/2}\, e^{-y(2-t)/2}\, dy = \left(\frac{1}{2\pi}\right)^{1/2} \frac{(2\pi)^{1/2}}{(2-t)^{1/2}}$$

for $2 > t$, as the integrand is proportional to a $Gamma(1/2, 2/(2-t))$ pdf. Hence

$$m_X(t) = \left(\frac{1}{1-2t}\right)^{1/2}$$

Now, let $U_1 = X_1^2$ and $U_2 = X_2^2$, so that U_1 and U_2 are independent $Gamma(1/2, 2)$ variables. Now we have from a key mgf result

$$V = U_1 + U_2 \quad\Longrightarrow\quad m_V(t) = m_{U_1}(t) m_{U_2}(t) = \left(\frac{1}{1-2t}\right)^{1/2} \left(\frac{1}{2-t}\right)^{1/2} = \left(\frac{1}{1-2t}\right)$$

and hence, noting that this is the mgf of a Gamma random variable with parameters 1 and 2, we conclude that

$$V \sim Gamma\,(1,2) = Gamma\left(\frac{2}{2}, 2\right) \equiv \chi_2^2$$